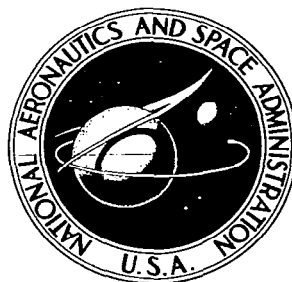


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**AN INVESTIGATION OF GROUND SHOCK  
EFFECTS DUE TO RAYLEIGH WAVES  
GENERATED BY SONIC BOOMS**

*by Melvin L. Baron, Hans H. Bleich, and Joseph P. Wright*

Prepared under Contract No. NAS 1-3477 by  
**PAUL WEIDLINGER, CONSULTING ENGINEER**  
New York, N. Y.  
*for Langley Research Center*

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# ABSTRACT

Expressions are derived for the steady state vertical displacements produced at the surface of an elastic half-space by a line load of finite length, which moves with a constant velocity in a direction either parallel or perpendicular to its length. These expressions are used to estimate the response of structures to the seismic disturbances produced by a sonic boom which moves at speeds close to the speed of surface waves in the medium. Shock amplification factors for the accelerations imparted to the structure are obtained for a range of parameters. The results show that the accelerations produced at these speeds are generally quite small and that the resonance peak which occurs when the applied load moves with the surface wave speed is extremely narrow.

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# LIST OF SYMBOLS<sup>\*)</sup>

$c_P, c_S, c_R$	Velocities of dilatational, equivoluminal and Rayleigh waves, respectively.
$C$	Constant.
$F$	Shock amplification factor, Eq. (II-9).
$k = \frac{T}{2\delta}$	Nondimensional parameter, Eq. (II-10).
$K$	Constant defined for stationary point load, see Appendix A, Eq. (1g).
$L$	Length of line load with space distribution perpendicular to its direction of propagation.
$P$	Observation point.
$P'$	Point of application of moving point load.
$r$	Distance, see Fig. A-1.
$R(t)$	Vertical surface displacement due to moving line load with a step distribution in time.
$R_\alpha$	Function, see Appendix A, Eq. (13).
$\bar{R}_\alpha$	Nondimensional function, see Appendix A, Eq. (13a).
$t$	Time.
$t'$	Time (integration variable).
$T = \frac{2\pi}{\omega}$	Period of structure.
$u_z(t), U_z(t)$	Vertical displacements due to moving line loads with space distribution parallel and perpendicular, respectively, to the direction of propagation.

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<sup>\*)</sup> Additional symbols are defined as they occur in the text.

$V'$	Speed of moving load.
$V = \frac{V'}{c_S}$	Mach number with respect to the equivoluminal velocity.
$w(\tau)$	Vertical displacement due to stationary point load.
$W(t)$	Vertical displacement due to moving point load.
$y = \frac{Y}{c_S}$	Time of travel of S-wave over distance Y.
$y(t)$	Displacement history of oscillator, Eq. (II-6).
$Y$	Y-coordinate of point P.
$Z$	Intensity of applied point load.
$\gamma = \frac{1}{2} \sqrt{3 + \sqrt{3}} = 1.08766... = \frac{c_S}{c_R} \text{ for } \nu = \frac{1}{4} .$ $\bar{\gamma} = \frac{1}{2} \sqrt{3 - \sqrt{3}} = 0.563016... .$	$\left. \begin{array}{l} \\ \end{array} \right\} \begin{array}{l} \text{Constants, see Appendix A,} \\ \text{Eqs. (1h).} \end{array}$
$\delta = \frac{L}{2V'}$	Time defining the length $2\delta V'$ of parallel line load.
$\mu$	Shear modulus.
$\omega$	Frequency of structure.
$\tau$	Nondimensional time.
$\xi$	Integration variable.
$\xi_0, \xi_1, \xi_Y, \xi_Y'$	End points of integration variable $\xi$ .

## I INTRODUCTION.

In anticipation of the development of supersonic transport planes, the question of potential shock effects in the ground due to sonic booms arises, i.e. shock effects on the foundations of surface structures or on shallow buried underground structures. The sonic boom produces a pressure wave over a crescent shaped area. The wave progresses along the ground with the speed  $V'$  of the plane and has an intended peak pressure of about two pounds per square foot. In general, shock effects from a pressure of this magnitude can be expected to be insignificant as confirmed by field observations. However, in both elastic solids [1] and in slightly dissipative solids [2], magnified shock phenomena are known to exist for certain specific velocities  $V'$  of the moving pressure wave, namely when  $V' = c_P$ ,  $c_S$  or  $c_R$ , the velocities of the dilatational, equivoluminal and Rayleigh waves in the medium on whose surface the pressure is applied. For structures which are located at or near the surface of the medium, the intensity of such shock effects is necessarily most severe in the case where  $V' = c_R$ , i.e. for Rayleigh waves where the energy is concentrated near the surface. For this reason the present report, which represents a preliminary study of sonic boom effects, considers the amplification of shock effects on surface structures for the case of an elastic solid when the velocity  $V'$  of the traveling pressure wave is equal to, or close to, the velocity  $c_R$  of Rayleigh waves in the medium. If such effects are of any practical significance, it is clear that they will be of particular importance in the medium where  $V'$  is approximately equal to  $c_R$  and where dissipation is not large. The present investigation represents an attempt to study the possible importance of these shock effects and to indicate the desirability and direction of further experimental and/or theoretical studies.

The analytical problem to be studied is the effect of a traveling crescent shaped pressure distribution at points at or near the surface of an elastic semi-infinite medium over which the wave moves. The most convenient starting point is a series of studies by Pekeris [3], who obtained closed form solutions for the surface displacements produced by a stationary concentrated point force suddenly applied to the surface of an semi-infinite elastic medium. The problem of a crescent shaped pressure distribution could be formulated by suitable integrations in space and time of the Pekeris solution. However, the double integrations inherent in such an approach represent a large and time consuming computer effort, whereas the physical significance of the practical problem can be judged from the solution of a series of considerably simpler problems (from a computational viewpoint) involving moving line loads on the surface of the half-space. Two such types of problems have been considered: 1) where the space distribution of the line load is parallel to the direction of propagation (Fig. 1A) and 2) where the space distribution of the line load is perpendicular to the direction of propagation (Fig. 1B). For each of these cases, the steady state solution for the vertical displacement  $u_z$  was obtained for points on the surface in closed form and is presented in Appendix B.

The transient response of an elastic medium to moving point loads was recently derived by Peyton [4] but, prior to the publication of this paper, the present authors obtained the solution for the steady state problem of a traveling point load which moves with constant velocity on the surface of an elastic half-space. These results, which are presented in Appendix A, were used to construct the traveling line load solutions in Appendix B.

Once this solution has been obtained a convenient method for judging the likelihood of damage on structures by ground shock phenomena must be determined.

This paper uses the "shock factor" approach which permits the study of the expected ground shock amplification effect as a function of  $\frac{V'}{c_R}$  by means of a single number. Such results are extremely useful in obtaining qualitative judgments of practical importance, as shown in Section II of this report. Shock factors for the case of the moving line load with a space distribution parallel to the direction of propagation have been evaluated for several cases of interest and have been used to draw conclusions on the physical importance of the sonic boom effects.

## II DISCUSSION OF RESULTS.

In order to understand the response of points on the surface of the medium due to sonic booms as the velocity  $V'$  approaches  $c_R$ , consider the line load treated as Case 1 of Appendix B.

For a line load of length  $L = 2\delta V'$  with a space distribution parallel to its direction of propagation, Fig. (1A), the vertical displacement  $u_z$  at points on the surface is given by the curves shown in Figs. (B-1) and (B-2) for an illustrative case with  $Y = 2\delta c_S$ .

Figure (B-1) shows results when  $V' \leq c_R$ . It is seen that as  $V'$  approaches  $c_R$  from below, a plateau of maximum displacement occurs over a distance whose width is of the order of the length of the traveling line load,  $2\delta V'$ . When  $V'$  equals  $c_R$ , however, the displacements are infinite over the entire range  $2\delta V'$ .

Figure (B-2) shows results for the range  $c_R < V' < c_S$  for the illustrative case where  $Y = 2\delta c_S$ . It is noted that the nature of the response curves for the vertical displacements is quite different from that obtained for  $V' \leq c_R$ . When  $c_R < V' < c_S$ , two infinite values of the displacement  $u_z$  are obtained at the points

$$\frac{c_S t}{Y} = \pm \frac{c_S \delta}{Y} + \frac{c_S}{V'} \sqrt{\frac{V'^2}{c_R^2} - 1} \quad (\text{II-1})$$

for all values of  $V'$ . These infinities in the displacement are a direct consequence of the fact that the traveling wave is "superseismic" with respect to Rayleigh waves in the media. These infinities are produced by a superposition of the infinite displacements which appear at  $\tau = \gamma$  in the influence functions  $w(\tau)$  for the stationary concentrated load, as derived by Pekeris, [3]. It is of interest to observe that there is no continuity in the response curves for

$V' < c_R$  and  $V' > c_R$ , Figs. (B-1)-(B-2), respectively. The point  $V' = c_R$  is a discontinuous point in the character of the solution.

The curves shown in Figs. (B-1)-(B-2) are typical displacements  $u_z$  for points which do not lie on the path of the moving line load, i.e. for points where  $Y$  is not equal to zero. If  $Y$  is equal to zero, the displacements  $u_z$  contain infinities for all values of the velocity  $V'$  at the instants when the beginning or end points of the traveling load pass over the target point.

If the line load is extended in the transverse ( $Y$ ) direction, so that the pressure is applied to a finite rectangular area, no substantial change in the displacement patterns shown in Fig. (B-1) would occur. In the displacement pattern of Fig. (B-2), however, the infinite peaks would vanish so that for  $V' \neq c_R$ , the displacements due to the rectangular area loading would be finite everywhere, as one would expect from physical considerations.

To illustrate this mathematically, we note that the response of the medium due to a loading applied over a moving rectangular area can be derived by an integration of the response  $u_z$ , due to a moving load with a space distribution parallel to the direction of propagation of the load. For a rectangular area of width  $b-a$  and length  $L = 2\delta V'$ , the response  $u_R$  is given by

$$u_R(y, \delta, t) = \frac{c_S}{b-a} \int_{a/c_S}^{b/c_S} u_z(y-\eta, \delta, t) d\eta \quad (\text{II-2})$$

where  $u_z$  is given by Eqs. (10) and (11) of Appendix B. The quantity  $u_z$  becomes infinite for the times  $t$  defined by Eq. (II-1), as shown in Fig. (B-2). However, upon substitution of  $u_z$  into Eq. (II-2), it is seen that the contribution of the logarithmic terms in the second and fifth lines of Eq. (11), Appendix B, which give rise to these infinities, cancel for the times  $t$  defined by Eq. (II-1), thus giving a finite result for the rectangular loading.

The response for the rectangular loading could also be obtained by an integration of the response  $U_z$ , due to a line load with a space distribution perpendicular to the direction of propagation of the load. For this case, a similar situation occurs with respect to cancellation of the infinities in  $U_z$ , Eq. (16), Appendix B, for  $V' > c_R$ . However, the response still becomes infinite as  $V'$  approaches  $c_R$ .

It is crucial that the infinite vertical surface displacement found for the line load as  $V'$  approaches  $c_R$  occurs also if the load is applied over a finite area. Consequently, the qualitative behavior of the elastic half-space to a sonic boom as  $V'$  approaches  $c_R$  can be judged from the analysis for the traveling line load. It should also be mentioned that the authors consider it obvious that the horizontal displacements will, in principle, behave in a manner similar to the vertical displacements. Consequently, it was judged unnecessary to compute the horizontal displacements. The latter contain elliptic integrals which lead to even more cumbersome integrations than in the case of the vertical displacements in Appendix (A) and (B).

It is important, however, to realize that infinite displacements are naturally a mathematical fiction, since the velocity  $V'$  of the boom cannot be steadily maintained at  $c_R$ , and since dissipation will necessarily be present in real materials. Consequently, the infinite displacements which are predicted by the idealized theoretical model will be reduced to finite, but possibly large values. Moreover, possible structural damage due to sonic boom phenomena does not depend on the displacements of the medium, but rather on the accelerations which are produced by the boom. The effects of the accelerations on structures must therefore be considered.

A convenient way to ~~assess~~ the potential shock damage is by means of shock amplification factors which express the accelerations to which a structure in

the path of the sonic boom would be subjected. To compute these factors, the displacement history  $u_z(t)$  may be used directly in the following manner.

Idealizing the structure as a simple oscillator, one can obtain the response of the oscillator of frequency  $\omega$  to a displacement history  $u_z(t)$  of its support, without having to compute the accelerations applied to the support. Let a concentrated mass  $M$  on a linear spring of constant  $K$  be attached to a support which is subjected to a time dependent excitation, Fig. (2). Let  $u_z(t)$  be the excitation of the support, and  $y(t)$  be the displacement of the mass  $M$ , both relative to a fixed datum. The equation of motion of the oscillator is

$$M\ddot{y} = -K(y - u_z) \quad (\text{II-3})$$

so that

$$\ddot{y} + \omega^2 y = \omega^2 u_z \quad (\text{II-4})$$

where the frequency  $\omega$  of the oscillator is

$$\omega = \sqrt{\frac{K}{M}}. \quad (\text{II-5})$$

For homogeneous initial conditions,  $y(0) = \dot{y}(0) = 0$ , the solution of Eq. (II-4) becomes

$$y(t) = \omega \int_0^t u_z(\tau) \sin \omega(t-\tau) d\tau. \quad (\text{II-6})$$

For a given frequency  $\omega$ , Eq. (II-6) gives the displacement history  $y(t)$  of the oscillator. The maximum displacement of the mass  $M$  relative to the moving support is given by

$$\left| y - u_z \right|_{\text{maximum}} \quad (\text{II-7})$$

while the peak absolute acceleration of the mass  $M$  is

$$\omega^2 \left| y - u_z \right|_{\text{maximum}}. \quad (\text{II-8})$$

Shock spectra giving the maximum displacements and accelerations of the mass  $M$  as a function of the frequency  $\omega$  may thus be computed.

The shock amplification factor, relative to a selected reference, is then defined by the ratio

$$F = \frac{\left[ \omega^2 |y - u_z|_{\max} \right]}{\left[ \omega^2 |y - u_z|_{\max} \right]_{\text{reference}}} \quad . \quad (\text{II-9})$$

To apply the shock factor concept to the present problem, one can study the magnification effects on the acceleration for typical values of the frequency  $\omega$ , as the ratio  $\frac{V'}{c_R}$  is varied [thus varying the excitation  $\omega^2 u_z$  in Eq. (II-4)]. The magnification effects are of particular interest as  $\frac{V'}{c_R}$  approaches unity. For this study, a reference point,  $\frac{V'}{c_R} = 0.75$ ,  $\frac{y}{\delta} = 2$ , was selected somewhat arbitrarily. It is far enough removed from the range of the critical velocities  $\left( \frac{V'}{c_R} \approx 1 \right)$  so that it might safely be assumed that the accelerations produced by a sonic boom traveling at a velocity  $V' = 0.75 c_R$  are insignificant.

Applying the shock factor concept, infinite exciting displacements  $u_z$  would produce infinite peak accelerations, since Eq. (II-8) contains  $u_z$ . The infinity would occur at the instant when  $u_z$  becomes infinite, that is at some instant or instants during the excitation period of the oscillator. When  $V' = c_R$ ,  $u_z$  becomes indeed infinite and the shock factor will therefore be infinite. For  $V' < c_R$ , Fig. (B-1) indicates no infinite values of  $u_z$ , so that the shock factors are everywhere finite. For  $V' > c_R$ , Fig. (B-2) for the line load indicates that  $u_z$  contains two infinities which consequently produce infinite shock factors during the excitation period. However, it has previously been pointed out that these infinities would not occur if the traveling pressure were realistically distributed over a finite area, i.e. that these infinities are solely due to the oversimplified approach of using a line load.

A typical plot of the acceleration  $\omega^2(y - u_z)$  versus time is given in Fig. (3) for  $\frac{V'}{c_R} = 1.05$ ,  $\frac{Y}{\delta} = 2$ . It illustrates the infinite peaks in the neighborhood of  $t = 0$  and shows that these spikes in the curve are extremely narrow. Recognizing that these spikes are only due to the replacement of the actual distributed load from a sonic boom by a line load, the maximum value of the acceleration elsewhere has been used in determining the shock factors which are listed in Table I.

Tables I(a)-(e) show the variation of the acceleration shock factor  $F$  as a function of  $\frac{V'}{c_R}$  for five values of the nondimensional parameter  $k$  and two values of  $\frac{Y}{\delta}$ . The parameter  $k$ , defined by

$$k = \frac{T}{2\delta}, \quad (\text{II-10})$$

relates  $T = \frac{2\pi}{\omega}$ , the period of the structure, to  $2\delta = \frac{L}{V'}$ , the time constant of the moving line load. The ratio  $\frac{Y}{\delta}$  relates  $Y = y c_S$ , the perpendicular distance from the point at which the structure is located to the line of propagation of the moving line load, to  $L = 2\delta V'$ , the length of the line load. Figures (4-A)-(4-E) present plots of the shock factors given in Tables I(a)-(e), respectively. Each figure corresponds to one of the values of  $k = 2, 6, 20, 60, 180$  and shows  $F$  as a function of  $\frac{V'}{c_R}$  for  $\frac{Y}{\delta} = 2$  and  $6$ . The parameters used in the computations cover the important ranges in building periods, Mach numbers and length of aircraft. Reasonable periods for buildings range from  $T = 0.2$  sec. to  $T = 6$  sec.

For a supersonic transport of length  $L = 300$  ft at Mach 3, the following range for the values of  $k$  is obtained:

$V' = 3,000$ ft/sec.	$T = 6$ sec.	;	$k = 60$
$L = 300$ ft	$T = 2$ sec.	;	$k = 20$
$2\delta = 0.1$	$T = 0.2$ sec.	;	$k = 2$

For a supersonic plane of length  $L = 100$  ft at Mach 3, the range of values for  $k$  becomes

$V' \doteq 3,000$ ft/sec.	$T = 6$ sec.	; $k = 180$
$L = 100$ ft	$T = 2$ sec.	; $k = 60$
$2\phi = 0.3333$	$T = 0.2$ sec.	; $k = 6$

For a supersonic plane of length  $L = 100$  ft at Mach 1, the range of values for  $k$  are the same as those for the  $L = 300$  ft, Mach 3 case.

It is seen that the increase in accelerations as  $\frac{V'}{c_R}$  approaches unity is restricted to an extremely narrow range of one or two percent on either side of  $\frac{V'}{c_R} = 1$ . This situation corresponds to a resonance with an extremely narrow peak, so that the presence of even the slightest amount of dissipation effectively eliminates the resonance for practical purposes. Moreover, one must also remember that the plane which produces a sonic boom can hardly do better than to maintain a speed  $V'$  which varies by one percent.

The largest values of the magnification factors  $F$  for  $\frac{V'}{c_R} = 1.01$  are in general below ten and only for the extreme case  $k = 2$  does the factor  $F$  reach forty. Available experimental information [5] has indicated that for  $V' \neq c_R$ , sonic boom effects transmitted through the ground are extremely small. Magnifications of ten are therefore clearly of no consequence. In the case  $k = 2$ , the magnification is forty, but reference to the parameters for the  $L = 300$  ft,  $M = 3$  case shows that this corresponds to a period  $T = 0.2$  sec., i.e. to very stiff buildings. In a stiff building, obviously a larger force can be tolerated without harmful effects.

### III CONCLUDING REMARKS.

It has been demonstrated in the paper that the potentially most serious situation occurs when the velocity of the plane is in close proximity to the speed of Rayleigh waves in the ground.

The paper lists magnification factors  $F$  which are functions of nondimensional parameters  $k$  and  $\frac{V}{c_R}$ . These amplification factors indicate the increase in all effects, (e.g. accelerations, stresses, etc.) when the velocity  $V'$  of the plane approaches the Rayleigh wave speed in the ground compared with a reference effect when  $V'$  differs substantially from  $c_R$ . It is known that this reference effect is extremely small; typical observations are given in [5].

The shock factors  $F$  were not computed for the crescent shaped signature of a sonic boom, but for a simplified loading assumption. These magnification factors may theoretically reach an infinite value for a perfectly elastic material when  $V'$  is exactly equal to  $c_R$ . The theoretical infinite values can not be reached in actuality because all real materials contain a damping mechanism<sup>\*)</sup> and no plane can in practice maintain exactly the critical velocity  $V' = c_R$  for the medium. The situation is quite similar to the response of a complicated mechanical oscillator subjected to a harmonic force when the frequency of the force coincides with one of the natural frequencies of the system. The practical importance of such a resonance depends on the narrowness of the spike in a diagram representing

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<sup>\*)</sup> The assumption of perfectly elastic behavior violates the laws of thermodynamics, unless the thermal expansion coefficient of the material were to vanish. A "corrected" analysis, using the equations of "thermo-elasticity", [6], contains a mechanism transferring minute portions of the mechanical energy into heat; i.e. there is slight damping. If the material should also be subject to viscous or plastic effects, the energy dissipation from these sources would be in addition to that from the thermal effects. The latter therefore gives the smallest damping effects for any real material.

amplitude as a function of frequency. In the present results the spike at  $V' = c_R$  is extremely narrow and the study therefore indicates that a rather delicate and very unlikely combination of circumstances would be required to produce significant responses.

A more definite answer, giving an upper bound on the effects, could be obtained on the basis of a thermo-elastic analysis. However, such an analysis would require a considerable effort in time and money. The writers are convinced that the resonance expressed by the infinite peak in the shock amplification factors, which exists for the ideal elastic model, can not be expected to be of practical significance. In view of this, they do not recommend continuation of analytical work on sonic boom effects transmitted to structures by surface waves in the ground.

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TABLE I - SHOCK FACTORS "F"

	F	
	k = 20	
$\frac{V'}{c_R}$	$\frac{Y}{\delta} = 2$	$\frac{Y}{\delta} = 6$
0.75	1.0	0.41
0.85	1.15	0.58
0.90	1.30	0.73
0.95	1.63	1.06
0.98	2.30	1.73
0.99	3.05	2.47
1.01	3.22	3.08
1.03	1.83	1.78
1.05	1.49	1.46
1.07	1.30	1.27

	F	
	k = 60	
$\frac{V'}{c_R}$	$\frac{Y}{\delta} = 2$	$\frac{Y}{\delta} = 6$
0.75	1.0	0.60
0.85	1.05	0.69
0.90	1.12	0.77
0.95	1.30	0.97
0.98	1.69	1.37
0.99	2.14	1.82
1.008	2.06	1.89
1.028	1.11	1.05
1.044	0.88	0.85
1.064	0.79	0.78

	F	
	k = 180	
$\frac{V'}{c_R}$	$\frac{Y}{\delta} = 2$	$\frac{Y}{\delta} = 6$
0.75	1.0	0.71
0.85	1.0	0.74
0.90	1.04	0.79
0.95	1.15	0.92
0.98	1.40	1.20
0.99	1.70	1.51
1.003	2.35	1.41
1.024	0.83	0.79
1.044	0.69	0.69

(a)

(b)

(c)

	F	
	k = 2	
$\frac{V'}{c_R}$	$\frac{Y}{\delta} = 2$	$\frac{Y}{\delta} = 6$
0.75	1.0	0.05
0.85	1.98	0.17
0.90	3.14	0.38
0.95	6.19	1.21
0.98	13.3	4.29
0.99	22.1	9.52
1.01	41.5	30.2
1.03	22.9	9.68
1.05	17.4	6.53
1.07	14.5	5.20

	F	
	k = 6	
$\frac{V'}{c_R}$	$\frac{Y}{\delta} = 2$	$\frac{Y}{\delta} = 6$
0.75	1.0	0.17
0.85	1.42	0.36
0.90	1.84	0.58
0.95	2.73	1.19
0.98	4.52	2.66
0.99	6.53	4.49
1.01	8.26	8.09
1.03	4.84	4.60
1.05	3.87	3.55
1.07	3.43	3.11

(d)

(e)

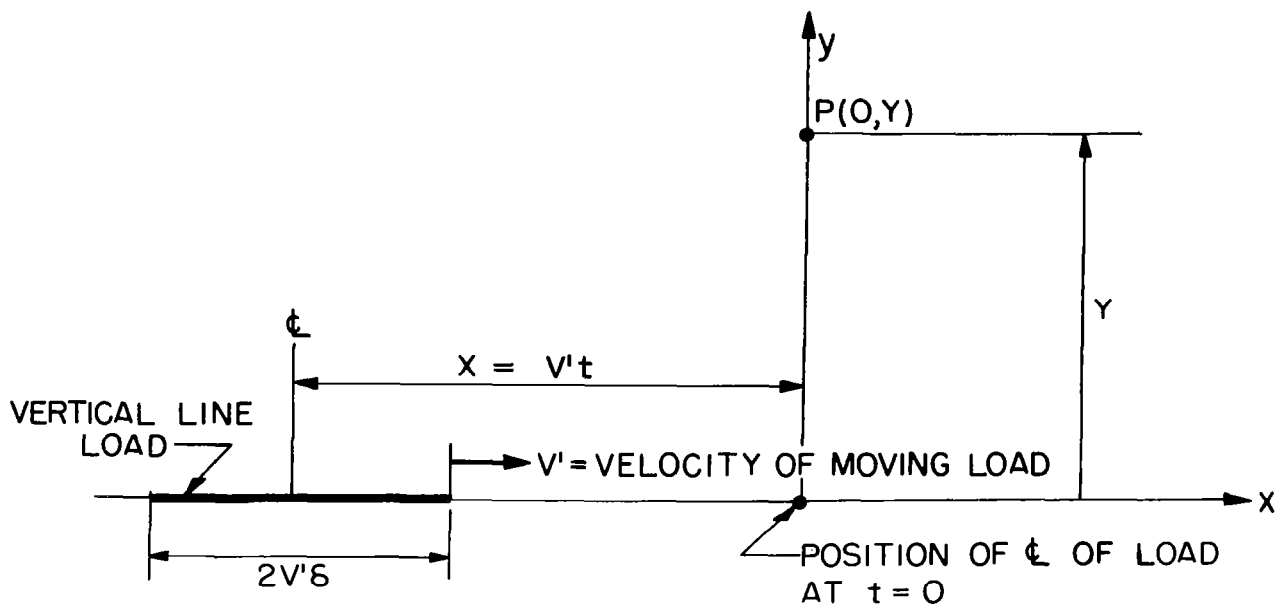


FIG. 1A PLAN VIEW OF MOVING LINE LOAD - TYPE I

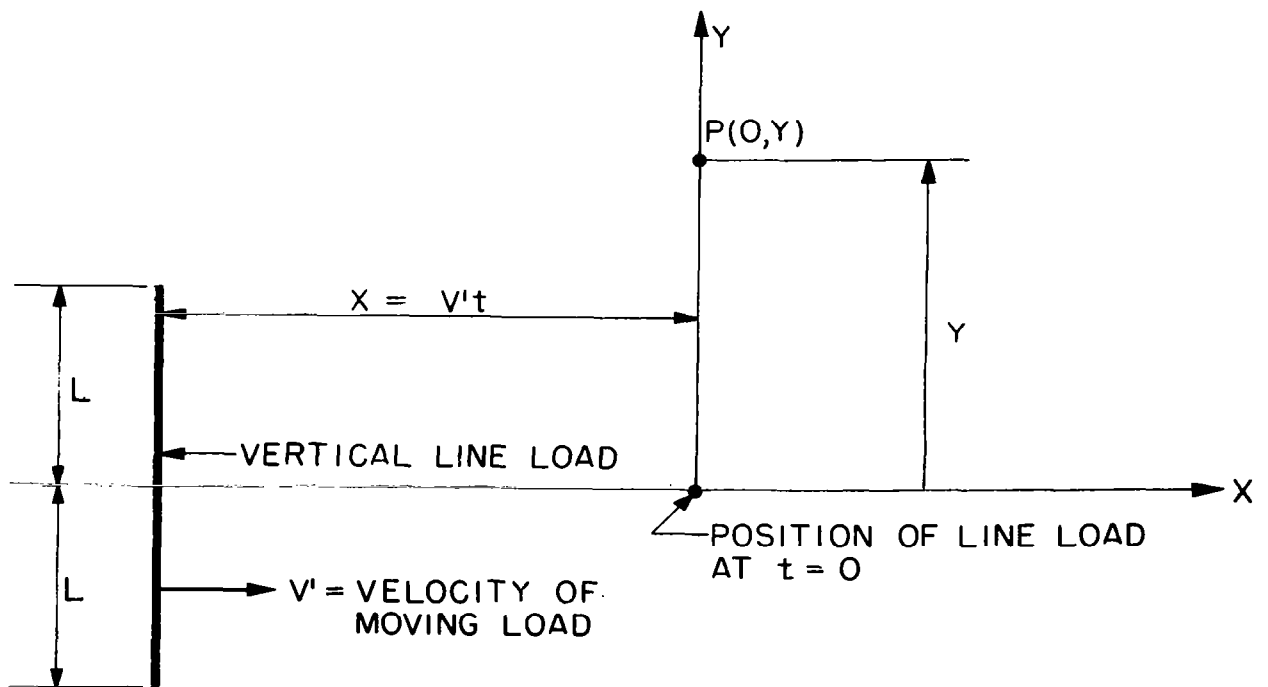


FIG. 1B PLAN VIEW OF MOVING LINE LOAD - TYPE II

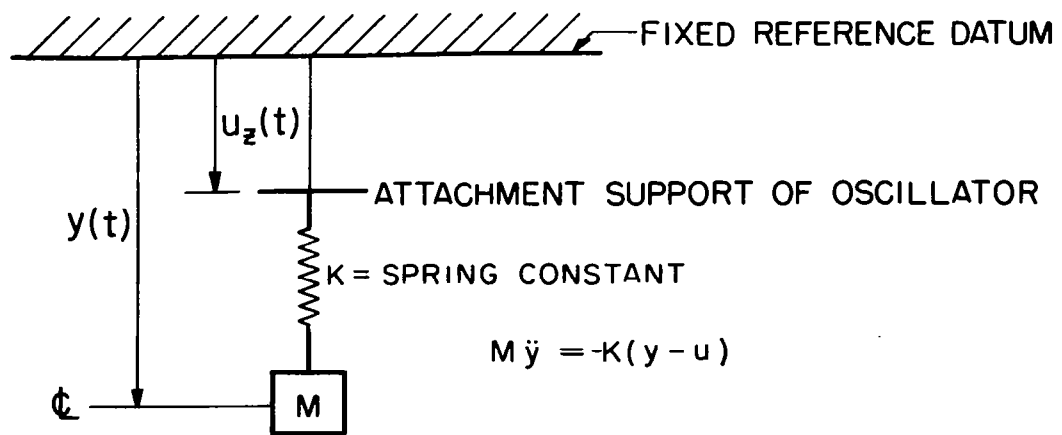


FIG. 2 LINEAR OSCILLATOR

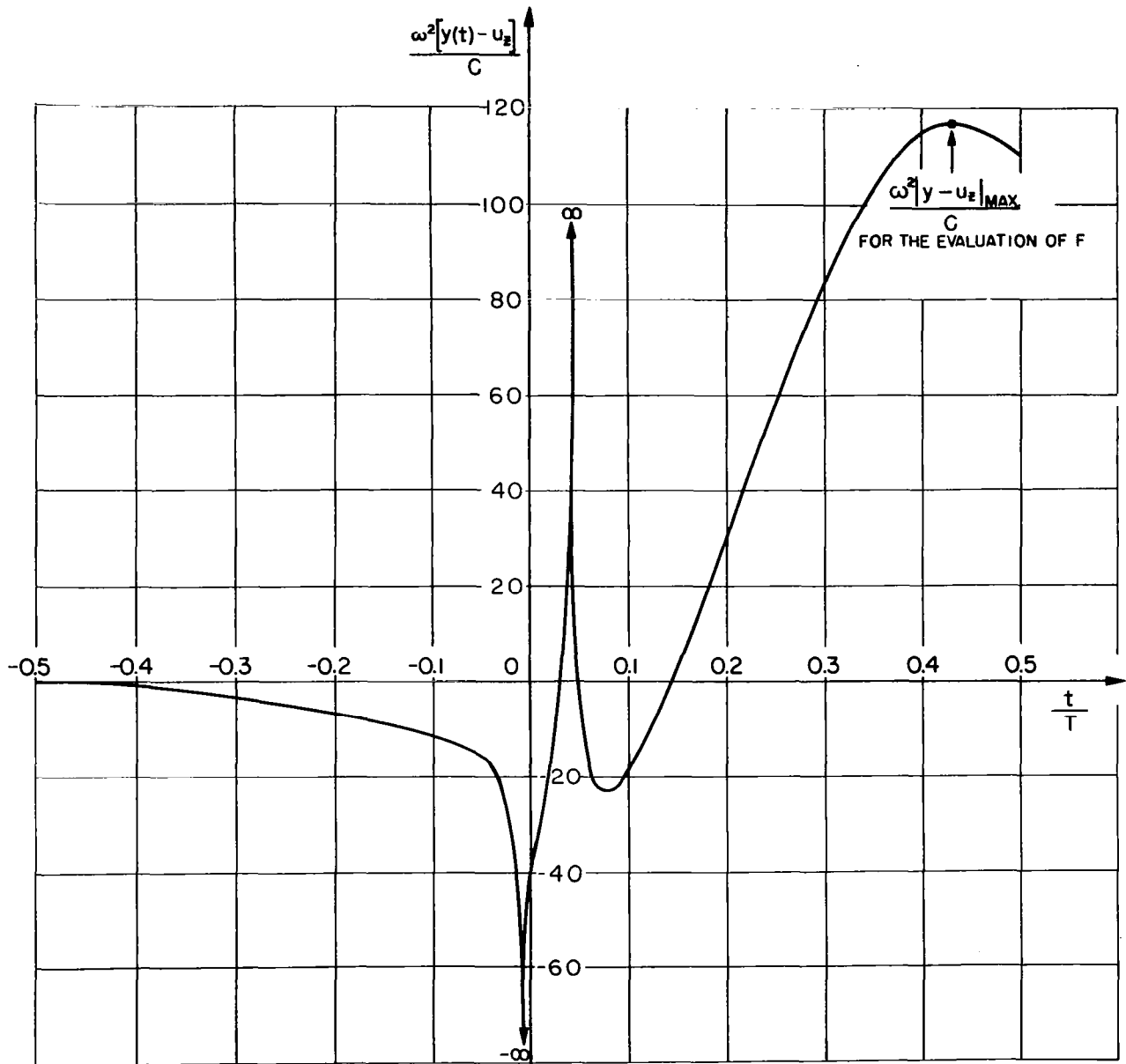
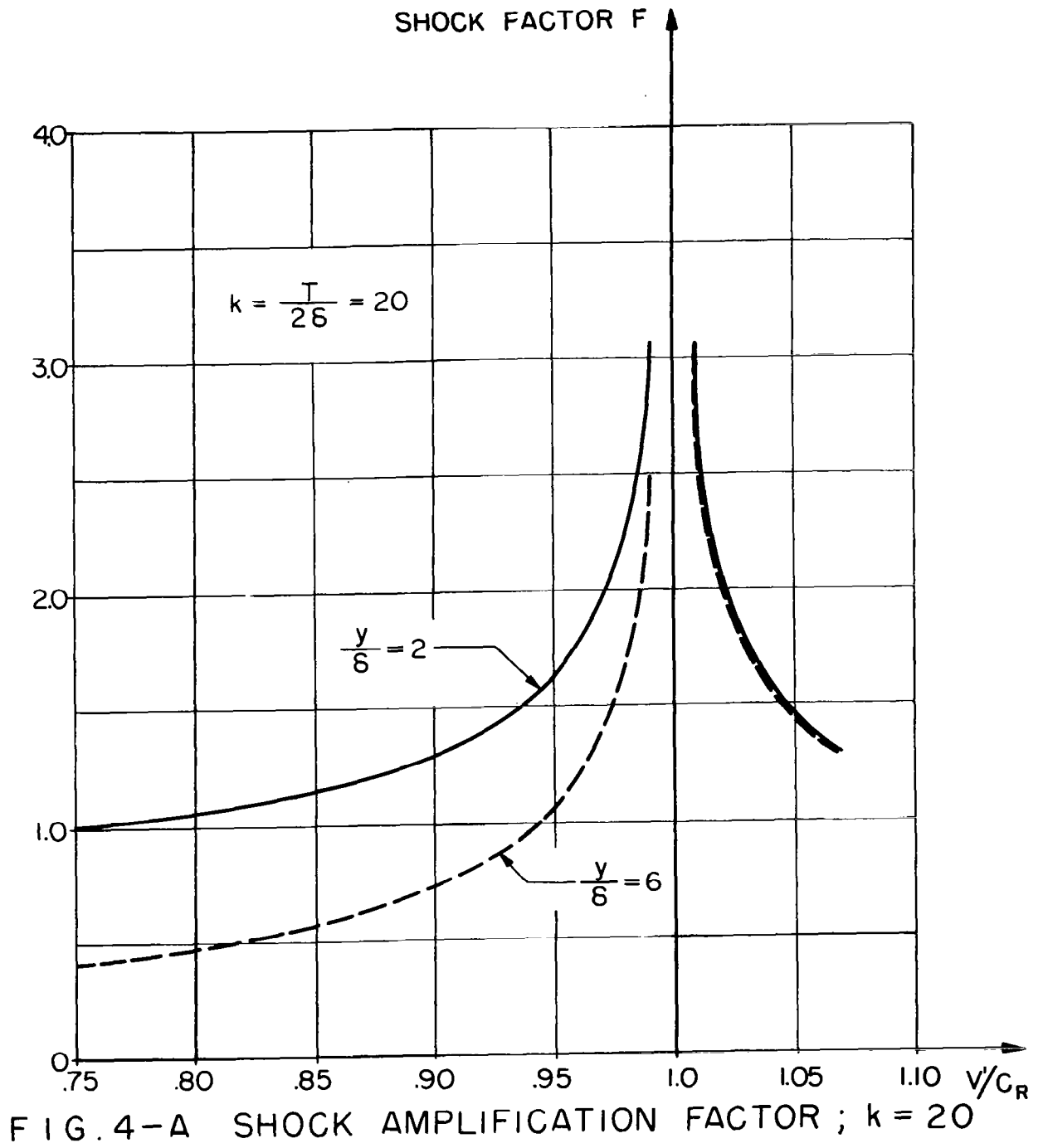


FIG.3—ACCELERATION VS. TIME FOR  $\frac{V'}{C_R} = 1.05$   
 $\frac{y}{\delta} = 2$  ;  $k = 2$



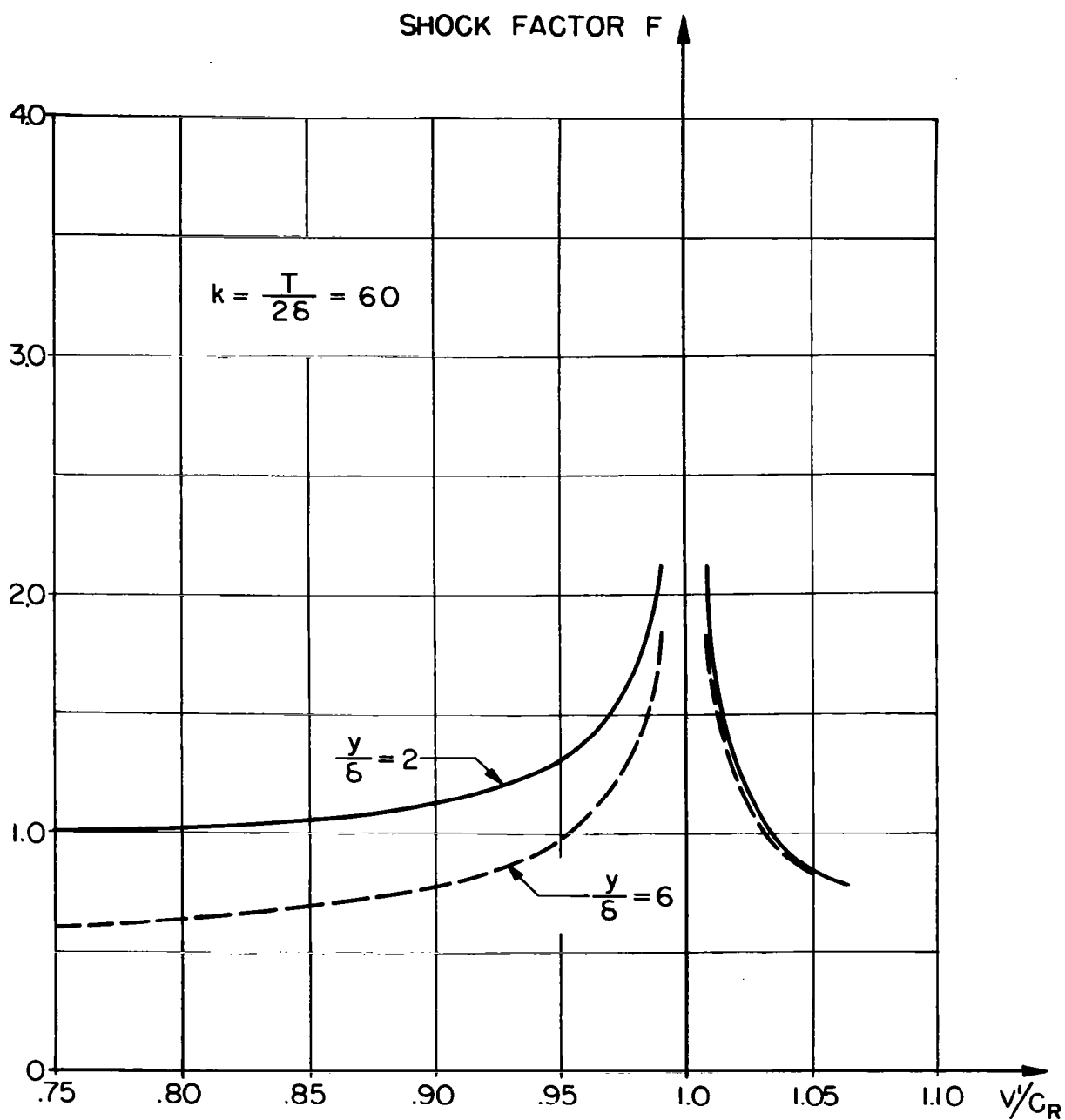


FIG. 4-B SHOCK AMPLIFICATION FACTOR ;  $k=60$

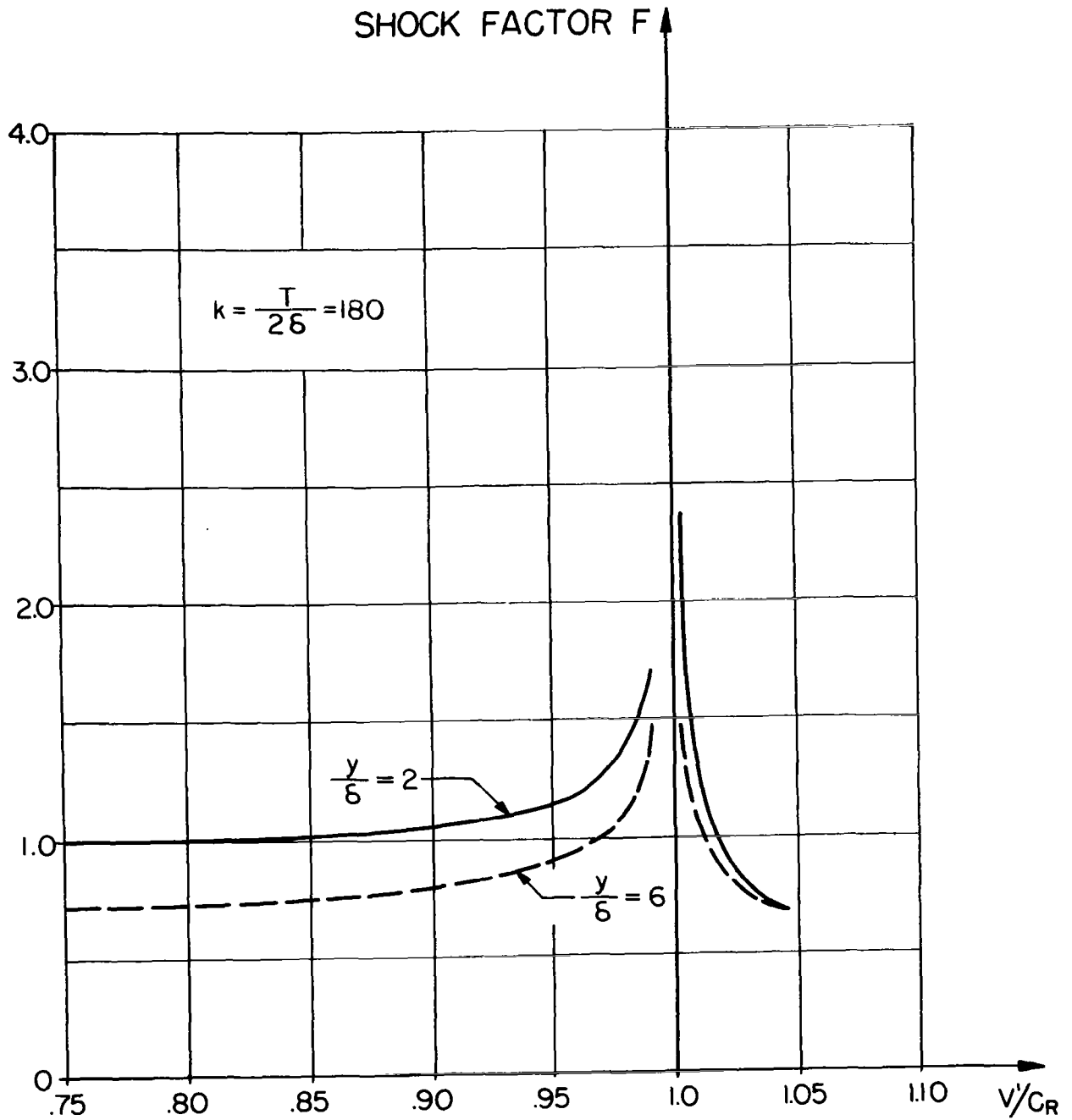


FIG. 4C — SHOCK AMPLIFICATION FACTOR ;  $k = 180$

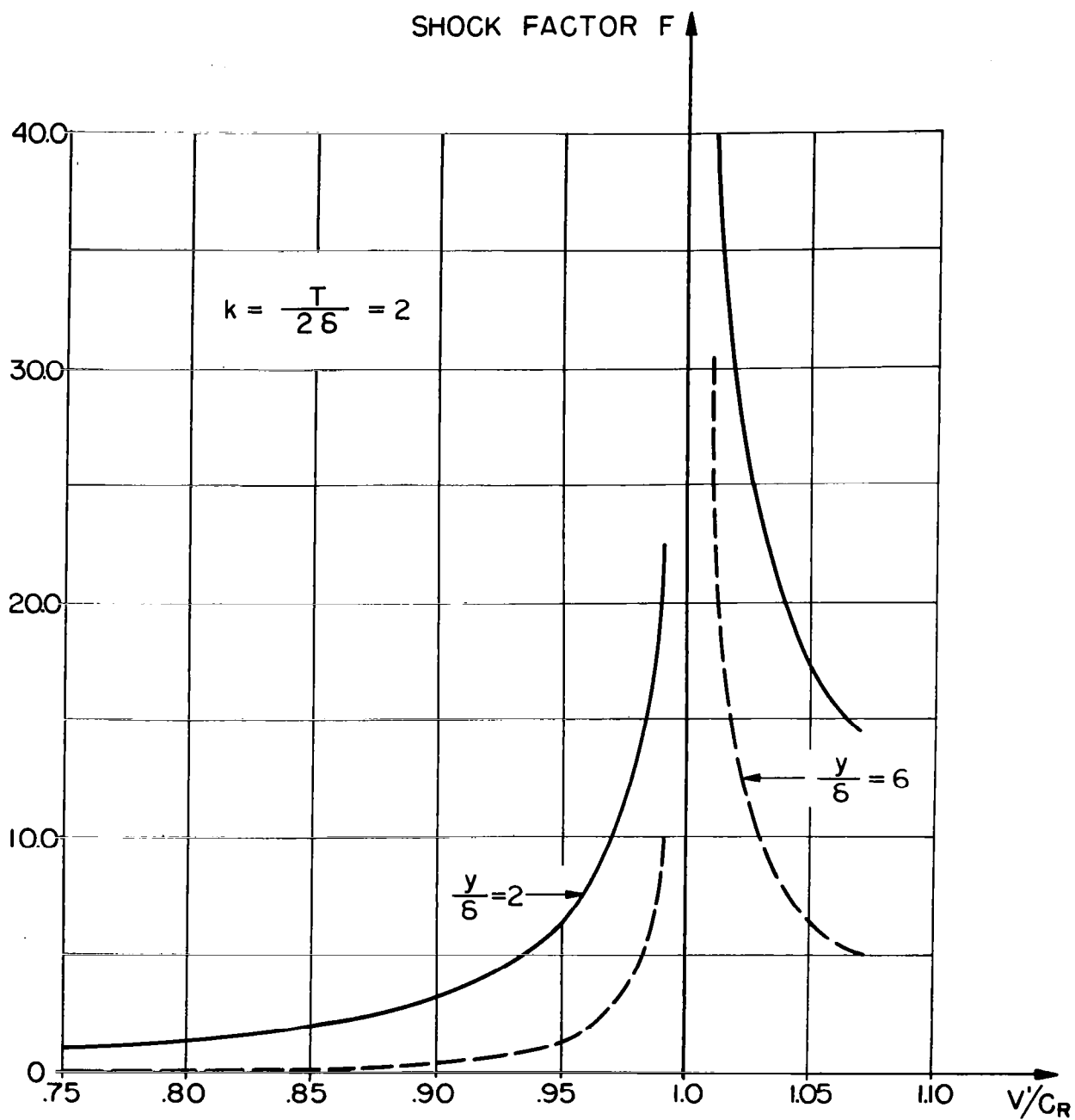


FIG. 4-D SHOCK AMPLIFICATION FACTOR ;  $k = 2$

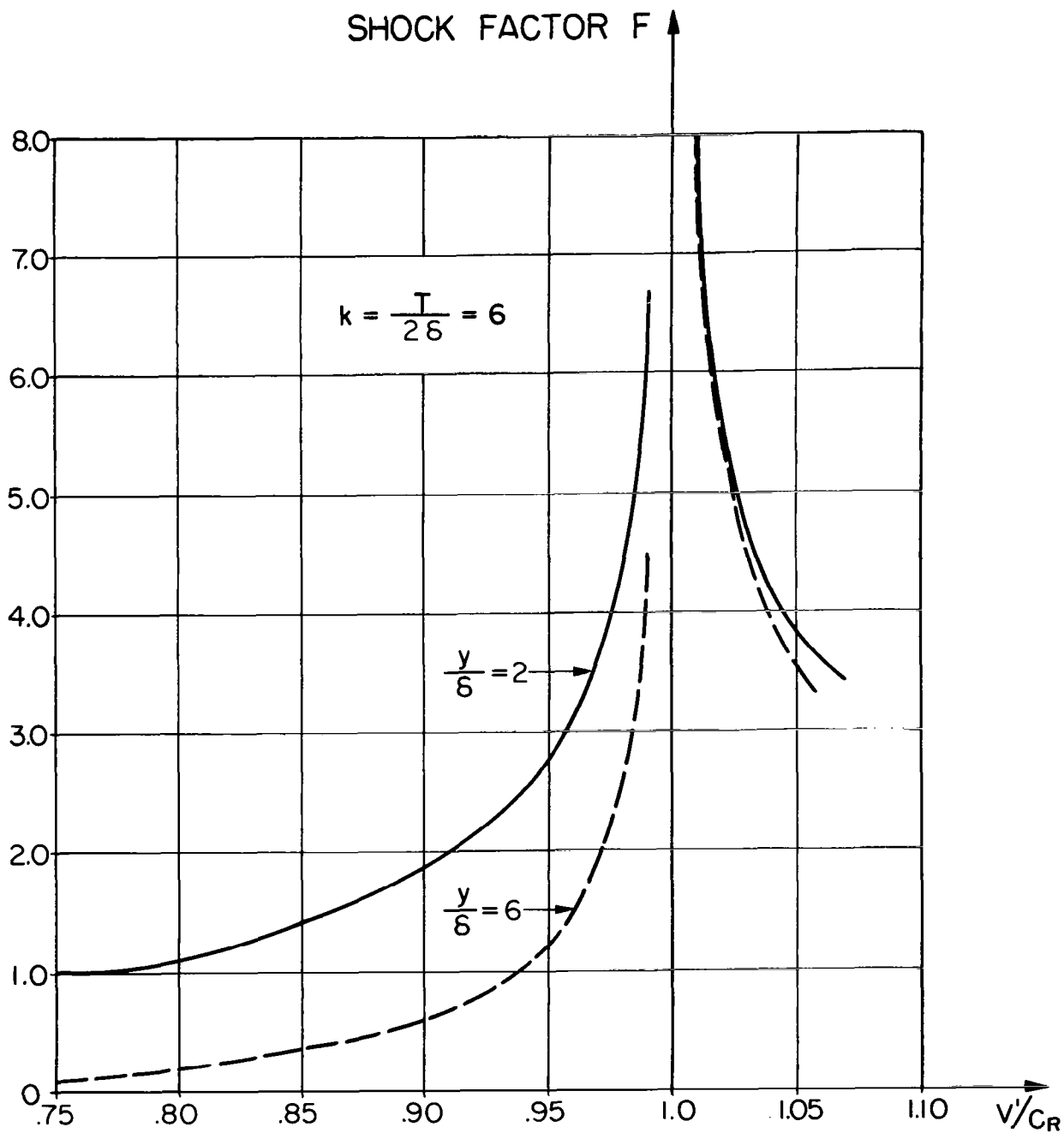


FIG. 4-E , SHOCK AMPLIFICATION FACTOR ;  $k = 6$

APPENDIX A - Steady State Response at the Surface of an Elastic Half-Space Due to Moving Point Load with Constant Velocity.

Consider a semi-infinite homogeneous and isotropic elastic half-space over which a concentrated point load moves with a velocity  $V'$ . The displacements at points on the surface of the medium can be evaluated by using the corresponding displacements due to a stationary point load as influence coefficients. The expressions for the displacements due to the moving load are evaluated by suitable integrations of the latter in time. This allows the study of the change in magnitude of the displacements as  $V'$ , the velocity of the point load is varied; of particular interest are the magnification effects as  $V'$  approaches  $c_R$ , the velocity of Rayleigh waves in the medium.

For practical purposes therefore, this study has been confined to an investigation of the vertical displacement  $W$  in the neighborhood of  $c_R$ . Results are presented for two regions: a)  $V' < c_R$  and b)  $c_S > V' > c_R$ .

The surface displacements produced on the half-space by a stationary concentrated point load with a step distribution in time have been obtained by Pekeris [3]\*). Using the nomenclature of Fig. (A-1), the vertical displacement  $w$  at a point  $P$ , due to the stationary load applied at a time  $t'$  and a point  $P'$  is given by:

$$w(\tau) = \frac{2K}{r} \left\{ \begin{array}{l} w(\tau) = 0 \quad \tau < \frac{1}{\sqrt{3}} \quad (1a) \\ 6 - \frac{\sqrt{3\sqrt{3}+5}}{\sqrt{\tau^2 - \frac{1}{3}}} - \frac{\sqrt{3}}{\sqrt{\tau^2 - \frac{1}{4}}} + \frac{\sqrt{3\sqrt{3}-5}}{\sqrt{\tau^2 - \tau^2}} \end{array} \right\} \quad \frac{1}{\sqrt{3}} < \tau < 1 \quad (1b)$$

---

\*) The Pekeris solution is given for an elastic material in which the Lamé constants are equal, i.e.  $\lambda = \mu$  and hence, Poisson's ratio  $\nu = \frac{1}{4}$ .

$$w(\tau) = \frac{4K}{r} \left\{ 6 - \frac{\sqrt{3\sqrt{3}+5}}{\sqrt{\gamma^2 - \tau^2}} \right\} \quad 1 < \tau < \gamma \quad (1c)$$

$$w(\tau) = \frac{24K}{r} \quad \tau \geq \gamma \quad (1d)$$

where

$$\tau = \frac{c_S(t - t')}{r} \quad (1e)$$

$$r = \overline{PP'} = \sqrt{Y^2 + (V't')^2} \quad (1f)$$

$$K = - \frac{Z}{32 \pi \mu} \quad (1g)$$

and

$$\gamma = \frac{1}{2} \sqrt{3 + \sqrt{3}} \quad , \quad \tilde{\gamma} = \frac{1}{2} \sqrt{3 - \sqrt{3}} \quad (1h)$$

The constant  $Z$  (negative) as defined by Pekeris represents the strength of the applied point load.

Consider now the steady state problem of a concentrated point load which moves along the surface of the medium with a constant velocity  $V'$ . For this steady state problem, the initial time of application of the load is defined as  $t = -\infty$  while  $t = 0$  is defined as the time at which the moving load passes opposite the point  $P$  at which the vertical displacement  $W$  is to be computed. The problem is formulated in two steps. First the vertical displacement  $R(t)$  due to a moving line load with a step function distribution in time is written in terms of the Pekeris solution, Eq. (1):

$$R(t) = \int_{-\infty}^t w(\tau) dt' \quad (2)$$

The desired solution  $W(t)$  for a moving point load is obtained by differentiating  $R(t)$  with respect to time:

$$W(t) = \frac{dR(t)}{dt} = \frac{\partial}{\partial t} \int_{-\infty}^t w(\tau) dt' \quad (3)$$

where

$$\tau = \frac{c_S(t - t')}{r} .$$

Care must be taken in differentiating the integral in Eq. (3), since the Pekeris displacement function  $w(\tau)$  is discontinuous at  $\tau = \gamma$ , while the derivative  $\frac{\partial w}{\partial t}(\tau)$  is discontinuous when  $\tau = \frac{1}{\sqrt{3}}$ , 1, and  $\gamma$  respectively. It is therefore necessary to separate the function  $R(t)$  into a sum of integrals, each of which is continuous and differentiable in the range of integration except at the point  $\tau = \gamma$ . To accomplish this, it is convenient to make the change in variables

$$\left. \begin{aligned} \xi &= t - t' & ; & & y &= \frac{Y}{c_S} \\ V &= \frac{V'}{c_S} \end{aligned} \right\} \quad (4)$$

so that

$$R(t) = \frac{1}{c_S} \int_0^{\infty} w(\tau) d\xi \quad (5)$$

where

$$\tau = \frac{c_S \xi}{r} = \frac{\xi}{\sqrt{y^2 + V^2(t - \xi)^2}} . \quad (6)$$

Considering the two cases of interest

$$\text{Case A: } V' < c_R ; \quad V < \frac{1}{\gamma}$$

$$\text{Case B: } c_R < V' < c_S ; \quad \frac{1}{\gamma} < V < 1$$

the variation of  $\tau$  as a function of  $\xi$  must be studied. Figures (A-2)-(A-3) show curves of  $\tau(\xi)$  for Cases A and B respectively. In each case,  $\tau$  goes to zero as  $\xi$  approaches zero, while  $\tau$  goes to  $\frac{1}{\gamma}$  as  $\xi$  approaches  $\infty$ . The range of integration in Eq. (5) limits  $\xi$  to positive values between 0 and  $\infty$ .

The point

$$\xi' = t + \frac{V^2}{V'^2 t} \quad (7)$$

corresponding to the maximum value of  $\tau$ , is positive only for positive values of  $t$ , i.e. for times at which the surface pressure has reached and passed opposite the point at which the displacement is being observed.

Hence, in Case A,  $\tau$  has no relative maximum for  $t < 0$  but increases monotonically to  $\frac{1}{\gamma}$ . For  $t > 0$ , a relative maximum point exists and is greater than  $\frac{1}{\gamma}$ . In Case B, three separate regions in time must be considered. Again  $\tau$  increases monotonically to  $\frac{1}{\gamma}$  for  $t < 0$ . For  $0 < t \leq \frac{V}{V'} \sqrt{V'^2 V^2 - 1}$ , a relative maximum is obtained between  $\tau = 1$  and  $\tau = \gamma$  while for  $t > \frac{V}{V'} \sqrt{V'^2 V^2 - 1}$ , the maximum value of  $\tau$  is greater than  $\tau = \gamma$ . The lines  $\tau = \frac{1}{\gamma}$ ,  $\tau = 1$  and  $\tau = \gamma$  divide the graphs into the four regions in which the function  $w(\tau)$  is defined.

Figures (A-2)-(A-3) allow the separation of the function  $R(t)$  so that the differentiation in Eq. (3) can be performed. To write the integral  $R(t)$  for any specific case, one follows the proper curve from  $\xi = 0$  to  $\xi = \infty$ , choosing the correct form of  $w(\tau)$  from Eqs. (1a)-(1d) as determined by the values of  $\tau$ . The problems are considered separately for Cases A and B.

Case A:  $V < \frac{1}{\gamma}$

From Fig. (A-2), it is noted that each of the lines  $\tau = \frac{1}{\sqrt{3}}$ ,  $\tau = 1$  and  $\tau = \gamma$  intersect the  $\tau(\xi)$  curve at one point only, both for  $t < 0$  and  $t > 0$ . Letting  $\xi_0$ ,  $\xi_1$  and  $\xi_\gamma$  be the values of  $\xi$  corresponding to  $\tau = \frac{1}{\sqrt{3}}$ , 1 and  $\gamma$  respectively,  $R(t)$  is written in terms of the Pekeris displacements:

$$R(t) = \frac{2K}{c_s} \int_{\xi_0}^{\xi_1} \left\{ \frac{6}{r} - \frac{\sqrt{3\sqrt{3}+5}}{\sqrt{\gamma^2 r^2 - \xi^2}} - \frac{\sqrt{3}}{\sqrt{\xi^2 - \frac{1}{4}r^2}} + \frac{\sqrt{3\sqrt{3}-5}}{\sqrt{\xi^2 - \gamma^2 r^2}} \right\} d\xi +$$

$$+ \frac{4K}{c_s} \int_{\xi_1}^{\xi_\gamma} \left\{ \frac{6}{r} - \frac{\sqrt{3\sqrt{3}+5}}{\sqrt{\gamma^2 r^2 - \xi^2}} \right\} d\xi + \frac{4K}{c_s} \int_{\xi_\gamma}^{\infty} \frac{6}{r} d\xi \quad (8)$$

where

$$r = \sqrt{y^2 + V^2(t-\xi)^2} \quad (9)$$

Differentiating Eq. (8) with respect to  $t$  and using the relations

$$\left. \begin{aligned} \frac{\partial}{\partial t} (\gamma^2 r^2 - \xi^2)^{-\frac{1}{2}} &= -\frac{\partial}{\partial \xi} (\gamma^2 r^2 - \xi^2)^{-\frac{1}{2}} + \xi (\gamma^2 r^2 - \xi^2)^{-3/2} \\ \frac{\partial}{\partial t} (\xi^2 - \frac{1}{4}r^2)^{-\frac{1}{2}} &= -\frac{\partial}{\partial \xi} (\xi^2 - \frac{1}{4}r^2)^{-\frac{1}{2}} - \xi (\xi^2 - \frac{1}{4}r^2)^{-3/2} \\ \frac{\partial}{\partial t} (\xi^2 - \gamma^2 r^2)^{-\frac{1}{2}} &= -\frac{\partial}{\partial \xi} (\xi^2 - \gamma^2 r^2)^{-\frac{1}{2}} - \xi (\xi^2 - \gamma^2 r^2)^{-3/2} \end{aligned} \right\} \quad (10)$$

the function  $\frac{\partial R}{\partial t}$  becomes

$$\frac{\partial R}{\partial t} = \frac{2K}{c_S} \left\{ \int_{\xi_0}^{\xi_1} \frac{\sqrt{3} \xi d\xi}{(\xi^2 - \frac{r^2}{4})^{3/2}} - \int_{\xi_0}^{\xi_1} \frac{\sqrt{3\sqrt{3} - 5} \xi d\xi}{(\xi^2 - \sqrt{3}r^2)^{3/2}} - \int_{\xi_0}^{\xi_Y} \frac{\sqrt{3\sqrt{3} + 5} \xi d\xi}{(\sqrt{3}r^2 - \xi^2)^{3/2}} - \right. \\ \left. - \int_{\xi_1}^{\xi_Y} \frac{\sqrt{3\sqrt{3} + 5} \xi d\xi}{(\sqrt{3}r^2 - \xi^2)^{3/2}} + \left[ \frac{2\sqrt{3\sqrt{3} + 5}}{(\sqrt{3}r^2 - \xi^2)^{1/2}} \right]_{\xi=\xi_Y} \left( 1 - \frac{d\xi_Y}{dt} \right) \right\} \quad (11)$$

The four integrals in Eq. (11) can be evaluated immediately. The fifth term is cancelled by the contributions from the upper limit  $\xi = \xi_Y$  of the last two integrals. Hence, the non-zero terms arise from the evaluation of the integrals at the points  $\xi = \xi_0, \xi_1$  respectively. Performing the indicated integrations,  $W(t)$  [Eq. (3)] becomes:

$$W(t) = \frac{4K}{c_S} \left\{ R_1 \left[ \frac{2 + \sqrt{3}}{R_Y^2} + \frac{2 - \sqrt{3}}{R_Y^2} - \frac{1}{R_{\frac{1}{2}}^2} \right] + R \frac{1}{\sqrt{3}} \left[ \frac{\sqrt{3}}{R_Y^2} - \frac{\sqrt{3}}{R_Y^2} + \frac{3}{R_{\frac{1}{2}}^2} \right] \right\} \quad (12)$$

where

$$R_\alpha = \sqrt{(1 - \alpha^2 v^2) y^2 + v^2 t^2} \quad (13)$$

and  $v < \frac{1}{Y}$ .

It is convenient to obtain results for the nondimensional function  $\frac{W(t)Y}{4K}$  in terms of the nondimensional time  $\frac{c_S t}{Y}$ . For this purpose, Eqs. (12)-(13) are rewritten in the form

$$\frac{W(t)Y}{4K} = \bar{R}_1 \left[ \frac{2 + \sqrt{3}}{\bar{R}_Y^2} + \frac{2 - \sqrt{3}}{\bar{R}_V^2} - \frac{1}{\bar{R}_{\frac{1}{2}}^2} \right] + \frac{\bar{R}}{\sqrt{3}} \left[ \frac{\sqrt{3}}{\bar{R}_Y^2} - \frac{\sqrt{3}}{\bar{R}_V^2} + \frac{3}{\bar{R}_{\frac{1}{2}}^2} \right] \quad (12a)$$

where

$$\bar{R}_\alpha = \sqrt{(1 - \alpha^2 V^2) + V^2 \left( \frac{c_s t}{Y} \right)^2} \quad . \quad (13a)$$

Figure (A-4) shows results for a range of values of  $\frac{V'}{c_R}$ . As  $V'$  approaches  $c_R$  from below, the magnitude of the vertical displacement  $W(t)$  is amplified and approaches infinity when  $V' = c_R$ . These results will be used as influence coefficients in Appendix B in evaluating the vertical displacements  $u_z$  produced at the surface of the medium by traveling line loads.

Case B:  $\frac{1}{Y} < V < 1$

From Fig. (A-3), it is noted that each of the lines  $\tau = \frac{1}{\sqrt{3}}$ ,  $\tau = 1$  and  $\tau = Y$  intersect the  $\tau(\xi)$  curve at the points  $\xi_0$ ,  $\xi_1$  and  $\xi_Y$  respectively for  $t \leq \frac{Y}{V} \sqrt{Y^2 V^2 - 1}$ . Hence:

$$\begin{aligned}
R(t) = & \frac{2K}{c_S} \int_{\xi_0}^{\xi_1} \left\{ \frac{6}{r} - \frac{\sqrt{3\sqrt{3}+5}}{\sqrt{\gamma^2 r^2 - \xi^2}} - \frac{\sqrt{3}}{\sqrt{\xi^2 - \frac{r^2}{4}}} + \frac{\sqrt{3\sqrt{3}-5}}{\sqrt{\xi^2 - \gamma^2 r^2}} \right\} d\xi + \\
& + \frac{4K}{c_S} \int_{\xi_1}^{\infty} \left\{ \frac{6}{r} - \frac{\sqrt{3\sqrt{3}+5}}{\sqrt{\gamma^2 r^2 - \xi^2}} \right\} d\xi
\end{aligned} \tag{14}$$

Upon rearranging terms, it may be shown that Eq. (14) differs from Eq. (8) by the single term:

$$- 2 \int_{\xi_Y}^{\infty} \frac{\sqrt{3\sqrt{3}+5}}{\sqrt{\gamma^2 r^2 - \xi^2}} d\xi \tag{15}$$

and hence,  $\frac{\partial R}{\partial t}$  is obtained by differentiating this term and adding it to Eq. (11). The displacement  $W(t)$  is given by:

$$\begin{aligned}
W(t) = & \frac{4K}{c_S} \left\{ R_1 \left[ \frac{2+\sqrt{3}}{R_Y^2} + \frac{2-\sqrt{3}}{R_Y^2} - \frac{1}{R_{\frac{1}{2}}^2} \right] + \right. \\
& + R \frac{1}{\sqrt{3}} \left[ \frac{\sqrt{3}}{R_Y^2} - \frac{\sqrt{3}}{R_Y^2} + \frac{3}{R_{\frac{1}{2}}^2} \right] + \\
& \left. + \frac{\sqrt{3\sqrt{3}+5}}{\sqrt{\gamma^2 r^2 - 1}} \left( \frac{v^2 t}{R_Y^2} \right) \right\} \\
& t \leq \frac{Y}{V} \sqrt{\gamma^2 v^2 - 1}
\end{aligned} \tag{16}$$

where  $R_Y$  is given by Eq. (13).

For those values of time for which  $t > \frac{Y}{V} \sqrt{\gamma^2 v^2 - 1}$ , Fig. (A-3) indicates that the line  $\tau = \gamma$  intersects the  $\tau(\xi)$  curve at two points. Letting these points be  $\xi_Y$  and  $\xi_Y'$  respectively,  $R(t)$  is written as:

$$\begin{aligned}
R(t) = & \frac{2K}{c_S} \int_{\xi_0}^{\xi_1} \left\{ \frac{6}{r} - \frac{\sqrt{3\sqrt{3}+5}}{\sqrt{v^2 r^2 - \xi^2}} - \frac{\sqrt{3}}{\sqrt{\xi^2 - \frac{1}{4}r^2}} + \frac{\sqrt{3\sqrt{3}-5}}{\sqrt{\xi^2 - v^2 r^2}} \right\} d\xi + \\
& + \frac{4K}{c_S} \int_{\xi_1}^{\xi_Y} \left\{ \frac{6}{r} - \frac{\sqrt{3\sqrt{3}+5}}{\sqrt{v^2 r^2 - \xi^2}} \right\} d\xi + \frac{4K}{c_S} \int_{\xi_Y}^{\xi_Y'} \frac{6}{r} d\xi + \\
& + \frac{4K}{c_S} \int_{\xi_Y'}^{\infty} \left\{ \frac{6}{r} - \frac{\sqrt{3\sqrt{3}+5}}{\sqrt{v^2 r^2 - \xi^2}} \right\} d\xi .
\end{aligned} \tag{17}$$

It may be shown that Eq. (17) differs from Eq. (8) by the single term

$$- 2 \int_{\xi_Y}^{\infty} \frac{\sqrt{3\sqrt{3}+5}}{\sqrt{v^2 r^2 - \xi^2}} d\xi . \tag{18}$$

Since the non-zero contribution to the derivative of this term comes only from the upper limit  $\xi = \infty$ , the expression for  $W$  for this case is identical with that given by Eq. (16). Hence, the vertical displacement  $W(t)$  is given by Eq. (16) for all values of  $t$ .

Proceeding as in Case (A), the nondimensional function  $\frac{W(t)Y}{4K}$  is plotted as a function of  $\frac{c_S t}{Y}$ . Equation (16) is rewritten in the form

$$\begin{aligned}
\frac{W(t)Y}{4K} = & \bar{R}_1 \left[ \frac{2+\sqrt{3}}{\bar{R}_Y^2} + \frac{2-\sqrt{3}}{\bar{R}_Y^2} - \frac{1}{\bar{R}_{\frac{1}{2}}^2} \right] + \bar{R} \frac{1}{\sqrt{3}} \left[ \frac{\sqrt{3}}{\bar{R}_Y^2} - \frac{\sqrt{3}}{\bar{R}_Y^2} + \frac{3}{\bar{R}_{\frac{1}{2}}^2} \right] + \\
& + \frac{\sqrt{3\sqrt{3}+5}}{\sqrt{v^2 - 1}} \left[ \frac{v^2 \frac{c_S t}{Y}}{\bar{R}_Y^2} \right]
\end{aligned} \tag{16a}$$

where  $\bar{R}_\alpha$  is given by Eq. (13a). Figure (A-5) shows results for two values of the ratio  $\frac{V'}{c_R}$ . It is seen that in each case, infinite discontinuities occurs in the displacements. These results will be used as influence coefficients in Appendix B in evaluating the vertical displacements  $u_z$  produced at the surface of the medium by traveling line loads.

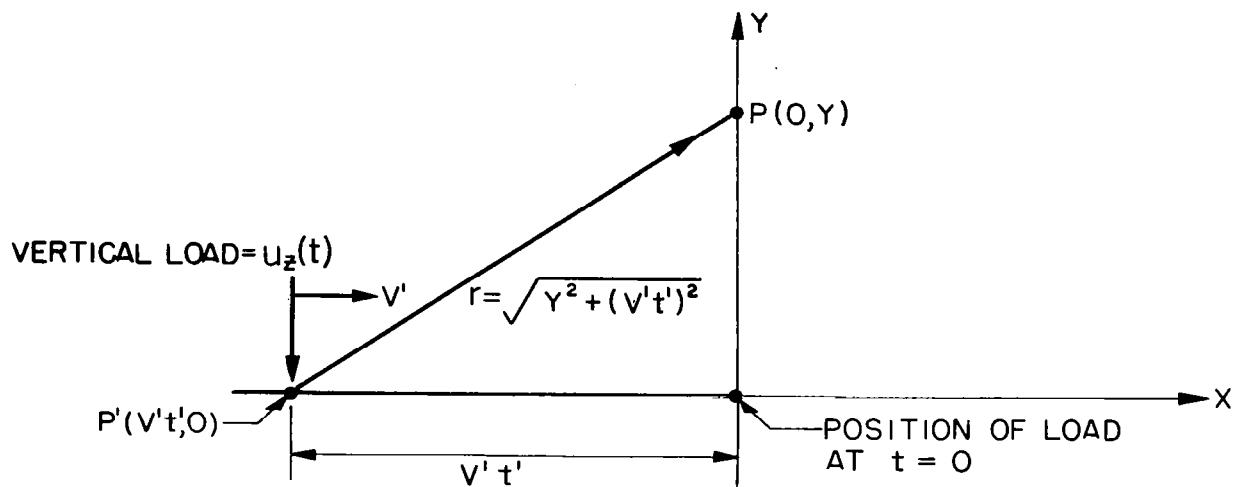


FIG. A-1 PLAN VIEW OF GEOMETRY FOR MOVING POINT LOAD PROBLEM

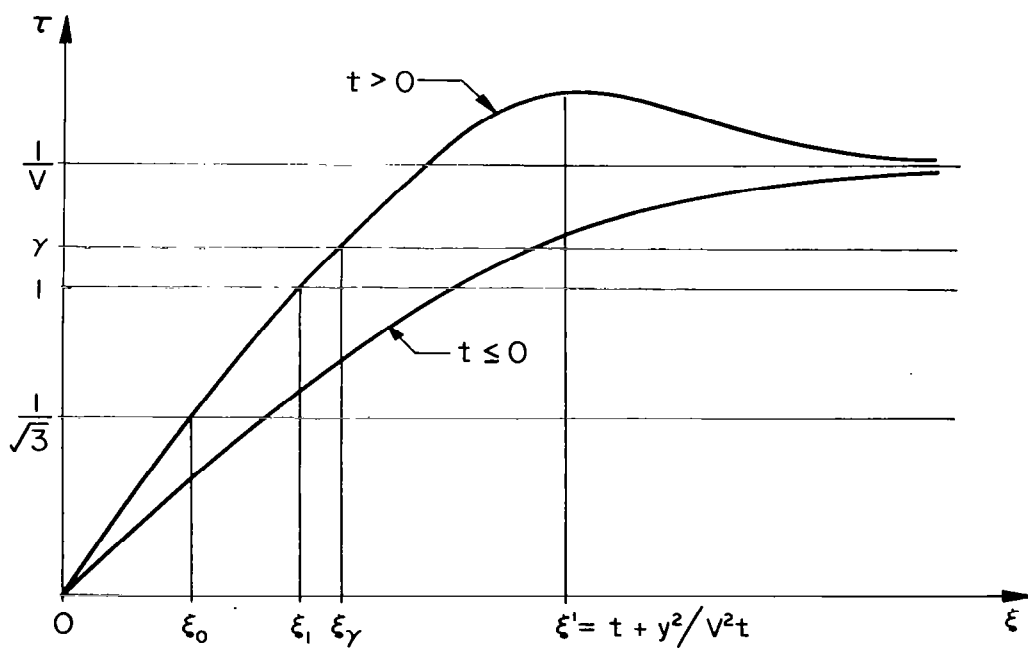


FIG. A-2  $\tau$  VS  $\xi$  CURVE FOR CASE A ;  $V < \frac{1}{\gamma}$

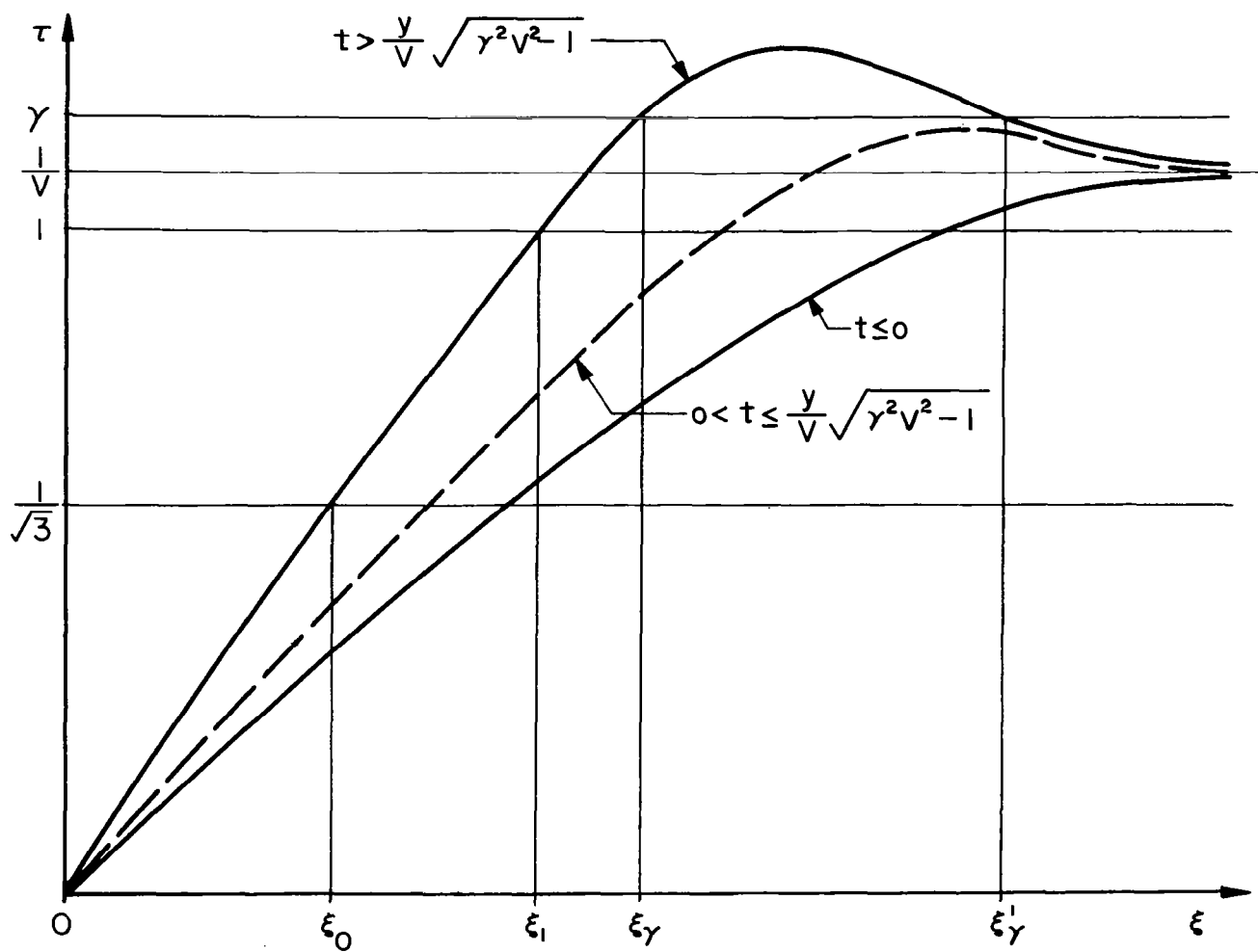


FIG. A-3  $\tau$  VS  $\xi$  CURVE FOR CASE B ;  $\frac{1}{\gamma} < V < 1$

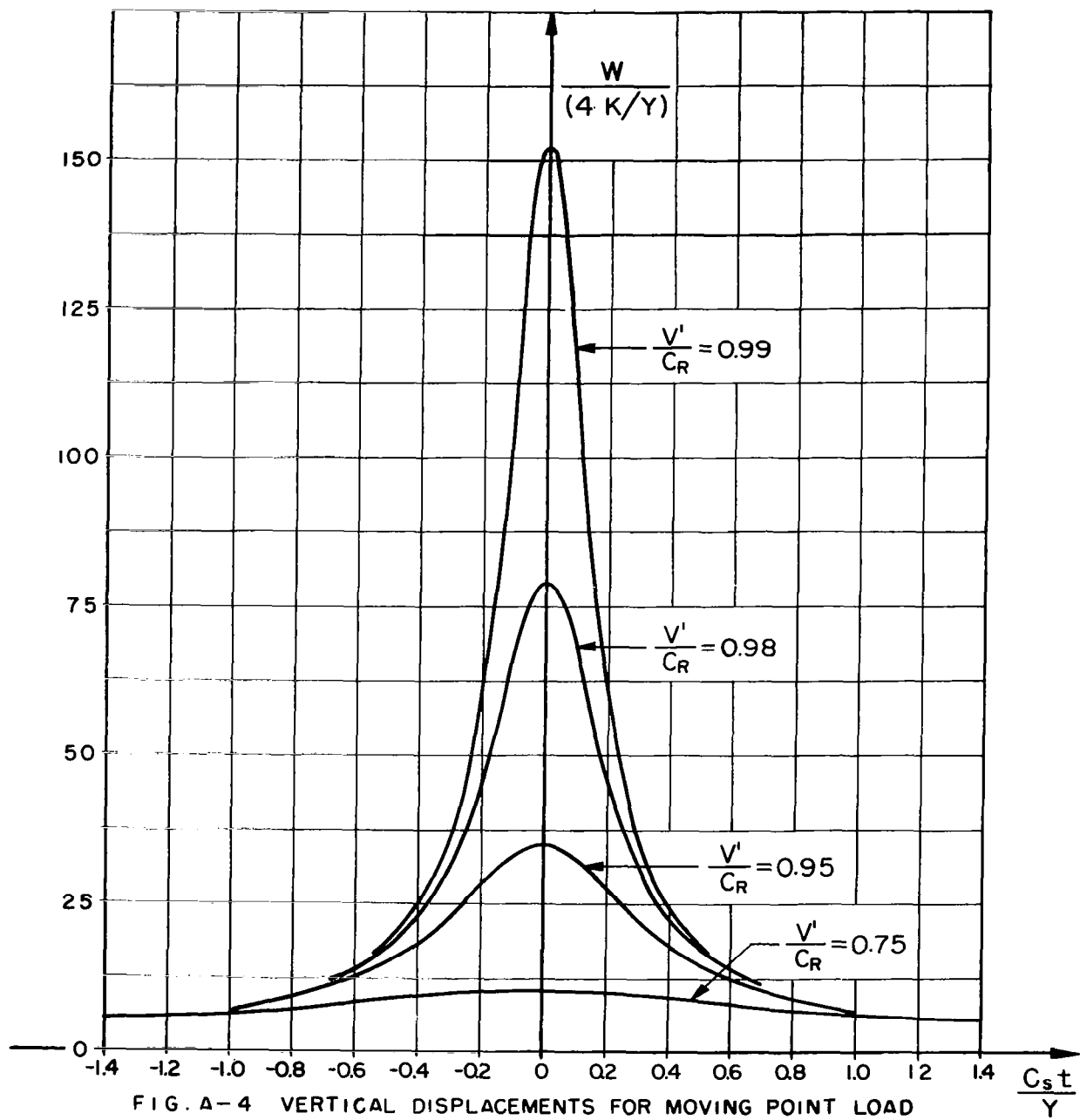


FIG. A-4 VERTICAL DISPLACEMENTS FOR MOVING POINT LOAD  
CASE A :  $V' < C_R$  ;  $\nu = 1/4$

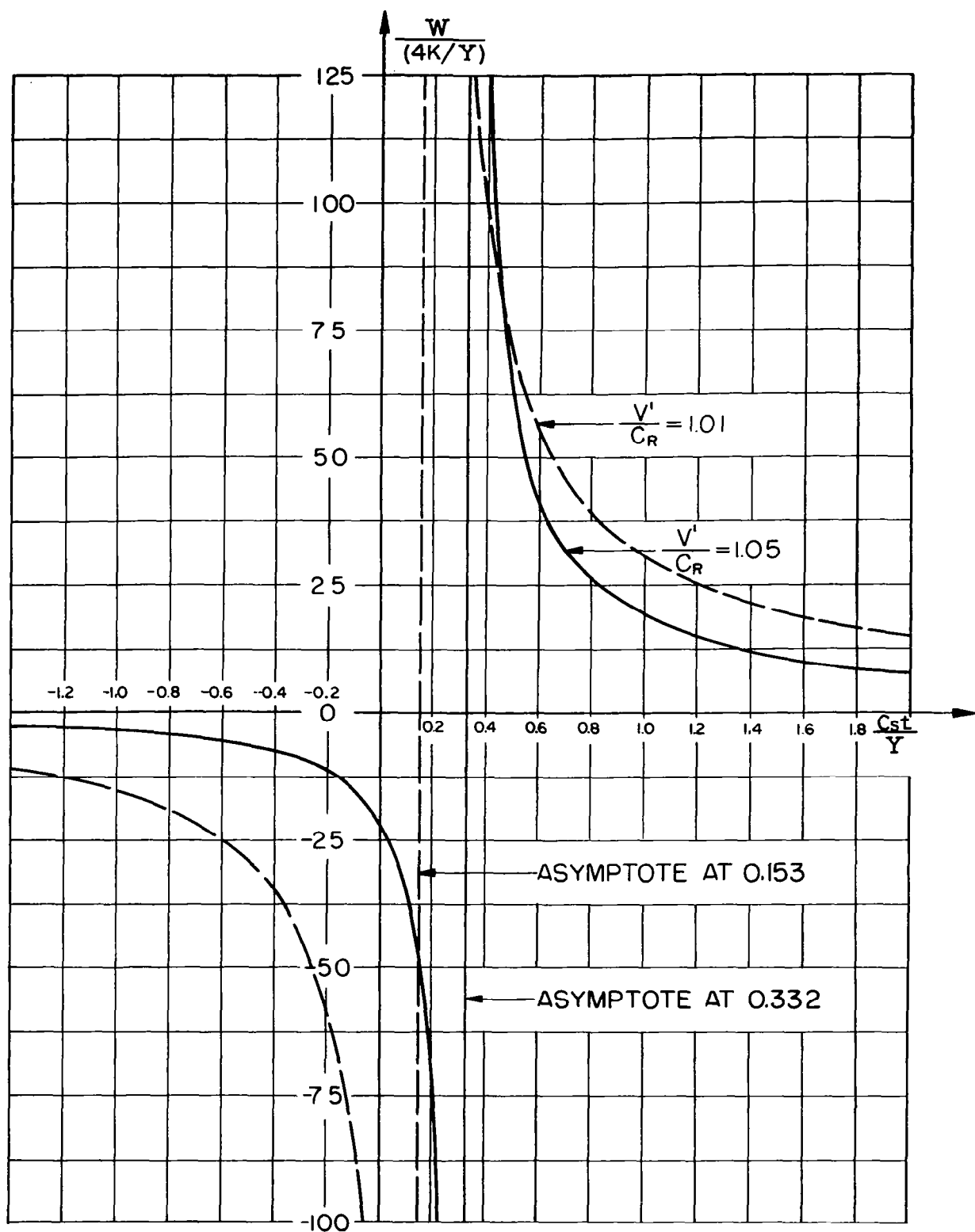


FIG. A-5 , VERTICAL DISPLACEMENT FOR MOVING POINT LOAD ; CASE  $B - C_R < V' < C_S$  ;  $\nu = 1/4$

APPENDIX B - Response at the Surface of an Elastic Half-Space Due to Line Loads Moving with Constant Velocity - Steady State Problems.

This Appendix presents the derivations of expressions for the vertical displacements produced at the surface of a semi-infinite elastic half-space by moving line loads with space distributions which are parallel or perpendicular, respectively, to the direction of propagation of the line loads. In each case, the solutions are obtained using the expressions for the vertical displacement  $W(t)$  which were derived in Appendix A for moving point loads. By means of a suitable integration in space, the corresponding displacements for the moving line loads are derived.

1. Moving Line Load with a Space Distribution Parallel to the Direction of Propagation of the Load [Fig. (1A)].

Consider a semi-infinite homogeneous and isotropic elastic half-space over which a line load of length  $2V'\delta$  moves with a constant velocity  $V'$ . The vertical displacement  $u_z$  at points on the surface of the medium can be evaluated by using the corresponding displacement  $W(t)$  due to a moving point load as obtained in Appendix A. As explained in the main body of the report, it was decided that it is sufficient to confine this study to an investigation of the vertical surface displacement  $u_z$  when  $V'$  is in the neighborhood of  $c_R$ . Results are presented for the two cases: A)  $V' < c_R$  and B)  $c_S > V' > c_R$ .

Consider the steady state problem of a line load of length  $2V'\delta$  which moves along the surface of the medium with a constant velocity  $V'$ . As in the case of the moving point load, the initial time of application of the

load is defined as  $t = -\infty$  while  $t = 0$  is defined as the time at which the center of the moving line load passes opposite the point P at which the vertical displacement  $u_z$  is to be computed. Referring to Fig. (1A), the displacement  $u_z$  is constructed by means of a space integration of  $W(V't, y)$ . The displacement  $du_z$  due to a differential element of the line load of length  $dx$  and unit intensity is given by the relation

$$du_z(y, t) = W(V't, y) dx \quad (1)$$

where  $x = V't$  is the position of the differential element and  $W(V't, y)$  is given by Eq. (12) of Appendix (A) for  $V' < c_R$ , and Eq. (16) of Appendix (A) for  $c_R < V' < c_S$ . The vertical displacement  $u_z(y, \delta, t)$  due to a moving line load of length  $2V'\delta$  and unit intensity is obtained by integration of Eq. (1):

$$u_z(y, \delta, t) = \frac{1}{2V'\delta} \int_{V'(t-\delta)}^{V'(t+\delta)} W(V't, y) dx = \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} W(V'\tau, y) d\tau \quad (2)$$

Case A:  $V' < c_R$ ,  $\left(V < \frac{1}{\gamma}\right)$

Substituting Eq. (12) of Appendix (A) into Eq. (2), the displacement  $u_z(y, \delta, t)$  is obtained directly in closed form by integration:

$$u_z(y, \delta, t) = \bar{u}_z(y, t+\delta) - \bar{u}_z(y, t-\delta) \quad (3)$$

where,

$$\begin{aligned}
\bar{u}_z(y, T) = \frac{2K}{8Vc_S} & \left[ 3 \ell n \left[ \left( T + \sqrt{T^2 + a^2} \right) \left( T + \sqrt{T^2 + b^2} \right) \right] + \right. \\
& + (2 + \sqrt{3}) \left( \frac{a^2 - l^2}{l^2} \right)^{\frac{1}{2}} \left\{ \arctan \left[ \left( \frac{a^2 - l^2}{l^2} \right)^{\frac{1}{2}} \frac{T}{\sqrt{T^2 + a^2}} \right] + \arctan \left[ \left( \frac{b^2 - l^2}{l^2} \right)^{\frac{1}{2}} \frac{T}{\sqrt{T^2 + b^2}} \right] \right\} + \\
& + \left( \frac{2 - \sqrt{3}}{2} \right) \left( \frac{m^2 - a^2}{m^2} \right)^{\frac{1}{2}} \ell n \left\{ \frac{\left[ 1 - \left( \frac{m^2 - a^2}{m^2} \right)^{\frac{1}{2}} \frac{T}{\sqrt{T^2 + a^2}} \right]}{\left[ 1 + \left( \frac{m^2 - a^2}{m^2} \right)^{\frac{1}{2}} \frac{T}{\sqrt{T^2 + a^2}} \right]} \left[ \frac{1 + \left( \frac{m^2 - b^2}{m^2} \right)^{\frac{1}{2}} \frac{T}{\sqrt{T^2 + b^2}} \right]}{\left[ 1 - \left( \frac{m^2 - b^2}{m^2} \right)^{\frac{1}{2}} \frac{T}{\sqrt{T^2 + b^2}} \right]} \right\} + \\
& - \frac{1}{2} \left( \frac{n^2 - a^2}{n^2} \right)^{\frac{1}{2}} \ell n \left\{ \frac{\left[ 1 - \left( \frac{n^2 - a^2}{n^2} \right)^{\frac{1}{2}} \frac{T}{\sqrt{T^2 + a^2}} \right]}{\left[ 1 + \left( \frac{n^2 - a^2}{n^2} \right)^{\frac{1}{2}} \frac{T}{\sqrt{T^2 + a^2}} \right]} \left[ \frac{1 + \left( \frac{n^2 - b^2}{n^2} \right)^{\frac{1}{2}} \frac{T}{\sqrt{T^2 + b^2}} \right]}{\left[ 1 - \left( \frac{n^2 - b^2}{n^2} \right)^{\frac{1}{2}} \frac{T}{\sqrt{T^2 + b^2}} \right]} \right\} \right] \quad (4)
\end{aligned}$$

and

$$\begin{aligned}
 a^2 &= \left[ \frac{1 - v^2}{v^2} \right] y^2 & ; & & b^2 &= \left[ \frac{1 - \frac{1}{3} v^2}{v^2} \right] y^2 \\
 l^2 &= \left[ \frac{1 - \frac{1}{2} v^2}{v^2} \right] y^2 & ; & & m^2 &= \left[ \frac{1 - \frac{1}{4} v^2}{v^2} \right] y^2 \\
 n^2 &= \left[ \frac{1 - \frac{1}{4} v^2}{v^2} \right] y^2 .
 \end{aligned} \tag{5}$$

It should be noted that the coefficient

$$\left[ \frac{a^2 - l^2}{l^2} \right] = \frac{v \sqrt{v^2 - 1}}{\sqrt{1 - \frac{1}{2} v^2}} \tag{6}$$

goes to infinity as  $v$  approaches  $\frac{1}{\sqrt{2}}$ , thus producing an infinite vertical surface displacement  $u_z(y, \delta, t)$ .

As an illustrative example, numerical results are shown for the case

$$\frac{\delta}{y} = \frac{c_S \delta}{Y} = 0.5 \tag{7}$$

$$v = \frac{1}{4} . \tag{8}$$

Figure (B-1) shows these results for a range of values of  $\frac{v'}{c_R}$ . These curves are discussed in Section II of the main body of the report.

#### Case B: $c_R < v' < c_S$

Substituting Eq. (16) of Appendix A into Eq. (2), the displacement  $u_z(y, \delta, t)$  is again obtained by integration. Since  $c_R < v' < c_S$  for this case, the arguments of the arctangent terms in Eq. (4) become imaginary and must be modified using the relation

$$\tan^{-1} ix = \frac{1}{2i} \ln \left[ \frac{1-x}{1+x} \right] \quad . \quad (9)$$

The expression for  $u_z(y, \delta, t)$  is

$$u_z(y, \delta, t) = \bar{u}_z(y, t+\delta) - \bar{u}_z(y, t-\delta) \quad (10)$$

where,

$$\begin{aligned}
\bar{u}_z(y, T) = & \frac{2K}{\delta v c_S} \left[ 3 \ell \ln \left[ (T + \sqrt{T^2 + a^2}) (T + \sqrt{T^2 + b^2}) \right] + \right. \\
& + \left( \frac{2 + \sqrt{3}}{2} \right) \sqrt{\frac{\ell^2 - a^2}{\ell^2}} \ell \ln \left\{ \left( \frac{1 - \sqrt{\frac{\ell^2 - a^2}{\ell^2}} \frac{T}{\sqrt{T^2 + a^2}}}{1 + \sqrt{\frac{\ell^2 - a^2}{\ell^2}} \frac{T}{\sqrt{T^2 + a^2}}} \right) \left( \frac{1 - \sqrt{\frac{\ell^2 - b^2}{\ell^2}} \frac{T}{\sqrt{T^2 + b^2}}}{1 + \sqrt{\frac{\ell^2 - b^2}{\ell^2}} \frac{T}{\sqrt{T^2 + b^2}}} \right) \right\} + \\
& + \left( \frac{2 - \sqrt{3}}{2} \right) \sqrt{\frac{m^2 - a^2}{m^2}} \ell \ln \left\{ \left( \frac{1 - \sqrt{\frac{m^2 - a^2}{m^2}} \frac{T}{\sqrt{T^2 + a^2}}}{1 + \sqrt{\frac{m^2 - a^2}{m^2}} \frac{T}{\sqrt{T^2 + a^2}}} \right) \left( \frac{1 + \sqrt{\frac{m^2 - b^2}{m^2}} \frac{T}{\sqrt{T^2 + b^2}}}{1 - \sqrt{\frac{m^2 - b^2}{m^2}} \frac{T}{\sqrt{T^2 + b^2}}} \right) \right\} - \\
& - \frac{1}{2} \sqrt{\frac{n^2 - a^2}{n^2}} \ell \ln \left\{ \left( \frac{1 - \sqrt{\frac{n^2 - a^2}{n^2}} \frac{T}{\sqrt{T^2 + a^2}}}{1 + \sqrt{\frac{n^2 - a^2}{n^2}} \frac{T}{\sqrt{T^2 + a^2}}} \right) \left( \frac{1 + \sqrt{\frac{n^2 - b^2}{n^2}} \frac{T}{\sqrt{T^2 + b^2}}}{1 - \sqrt{\frac{n^2 - b^2}{n^2}} \frac{T}{\sqrt{T^2 + b^2}}} \right) \right\} + \\
& + (2 + \sqrt{3}) \sqrt{\frac{\ell^2 - a^2}{\ell^2}} \ell \ln \left| T^2 + \ell^2 \right| \left. \right] \tag{11}
\end{aligned}$$

and  $a^2$ ,  $b^2$ ,  $l^2$ ,  $m^2$  and  $n^2$  are given by Eqs. (5).

Results are given again for the nondimensional function  $\frac{u_z Y}{4K}$  in terms of the nondimensional variable  $\frac{c_S t}{Y}$ . Figure (B-2) shows numerical results for the illustrative example,

$$\frac{\delta}{Y} = \frac{c_S \delta}{Y} = 0.5, \quad v = \frac{1}{4}$$

for two values of  $\frac{v'}{c_R}$  in the range  $c_R < v' < c_S$ .

These curves are discussed in Section II of the main body of the report.

## 2. Moving Line Load with a Space Distribution Perpendicular to the Direction of Propagation of the Load [Fig. (B-3)].

Consider a semi-infinite homogeneous and isotropic elastic half-space over which a line load of length  $2L$  moves with a constant velocity  $v'$ . The vertical displacement  $U_z$  at points on the surface of the medium is obtained by an integration procedure which is analogous to that used for the parallel line load.

Just as in the case of the point load, the initial time of application of the load is defined as  $t = -\infty$  while  $t = 0$  is defined as the time at which the moving line load passes opposite the point  $P$  at which the vertical displacement  $U_z$  is to be computed. Referring to Fig. (B-3), the displacement  $U_z$  is constructed by means of a space integration of  $W(v't, Y-\eta)$ . The displacement  $dU_z$  due to a differential element of line load of length  $d\eta$  and unit intensity is

$$dU_z(v't, Y-\eta) = W(v't, Y-\eta) d\eta \quad (12)$$

where  $W(V't, Y-\eta)$  is given by Eq. (12) of Appendix (A) for  $V' < c_R$ , and Eq. (16) of Appendix (A) for  $c_R < V' < c_S$ . The vertical displacement  $U_z(y, L, t)$  due to a moving line load of length  $2L$  is then obtained by integration of Eq. (12),

$$U_z(y, L, t) = \frac{1}{2L} \int_{-L}^L W(V't, Y-\eta) d\eta = \frac{c_S}{2L} \int_{y-L/c_S}^{y+L/c_S} W(V't, \zeta) d\zeta \quad (13)$$

Case A:  $V' < c_R$ ,  $\left(V < \frac{1}{\gamma}\right)$

Substituting Eq. (12) of Appendix (A) into Eq. (13), the displacement  $U_z(y, L, t)$  is obtained directly in closed form by integration:

$$U_z(y, L, t) = \bar{U}_z\left(y + \frac{L}{c_S}, t\right) - \bar{U}_z\left(y - \frac{L}{c_S}, t\right) \quad (14)$$

where,

$$\begin{aligned}
\bar{U}_z(y, t) = & \frac{2K}{VL} y \left[ a \left( \frac{2 + \sqrt{3}}{l^2} + \frac{2 - \sqrt{3}}{m^2} - \frac{1}{n^2} \right) \ln \left( a + \sqrt{t^2 + a^2} \right) + b \left( \frac{\sqrt{3}}{l^2} - \frac{\sqrt{3}}{m^2} + \frac{3}{n^2} \right) \ln \left( b + \sqrt{t^2 + b^2} \right) + \right. \\
& + \left( \frac{2 + \sqrt{3}}{2} \right) \frac{\sqrt{a^2 - l^2}}{l^2} \ln \left\{ \left( \frac{1 - \sqrt{a^2 - l^2} \frac{1}{\sqrt{t^2 + a^2}}}{1 + \sqrt{a^2 - l^2} \frac{1}{\sqrt{t^2 + a^2}}} \right) \left( \frac{1 - \sqrt{b^2 - l^2} \frac{1}{\sqrt{t^2 + b^2}}}{1 + \sqrt{b^2 - l^2} \frac{1}{\sqrt{t^2 + b^2}}} \right) \right\} + \\
& + (2 - \sqrt{3}) \frac{\sqrt{m^2 - a^2}}{m^2} \left\{ \arctan \left( \sqrt{m^2 - a^2} \frac{1}{\sqrt{t^2 + a^2}} \right) - \arctan \left( \sqrt{m^2 - b^2} \frac{1}{\sqrt{t^2 + b^2}} \right) \right\} - \\
& - \frac{\sqrt{n^2 - a^2}}{n^2} \left\{ \arctan \left( \sqrt{n^2 - a^2} \frac{1}{\sqrt{t^2 + a^2}} \right) - \arctan \left( \sqrt{n^2 - b^2} \frac{1}{\sqrt{t^2 + b^2}} \right) \right\} \Big]
\end{aligned} \tag{15}$$

and  $a^2$ ,  $b^2$ ,  $l^2$ ,  $m^2$  and  $n^2$  are given by Eq. (8).

Case B:  $c_R < V' < c_S$ ,  $\left(\frac{1}{\gamma} < V < 1\right)$

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Substituting Eq. (16) of Appendix (A) into Eq. (13), the displacement  $U_z(y, L, t)$  is again obtained by integration. The result is identical with Eq. (15) except for an additional term

$$(2 + \sqrt{3}) \frac{\sqrt{a^2 - l^2}}{l^2} \ln \left( \frac{1 - \sqrt{\gamma^2 V^2 - 1} \frac{y}{Vt}}{1 + \sqrt{\gamma^2 V^2 - 1} \frac{y}{Vt}} \right) \quad (16)$$

to be included inside the brackets, [...], of Eq. (15).

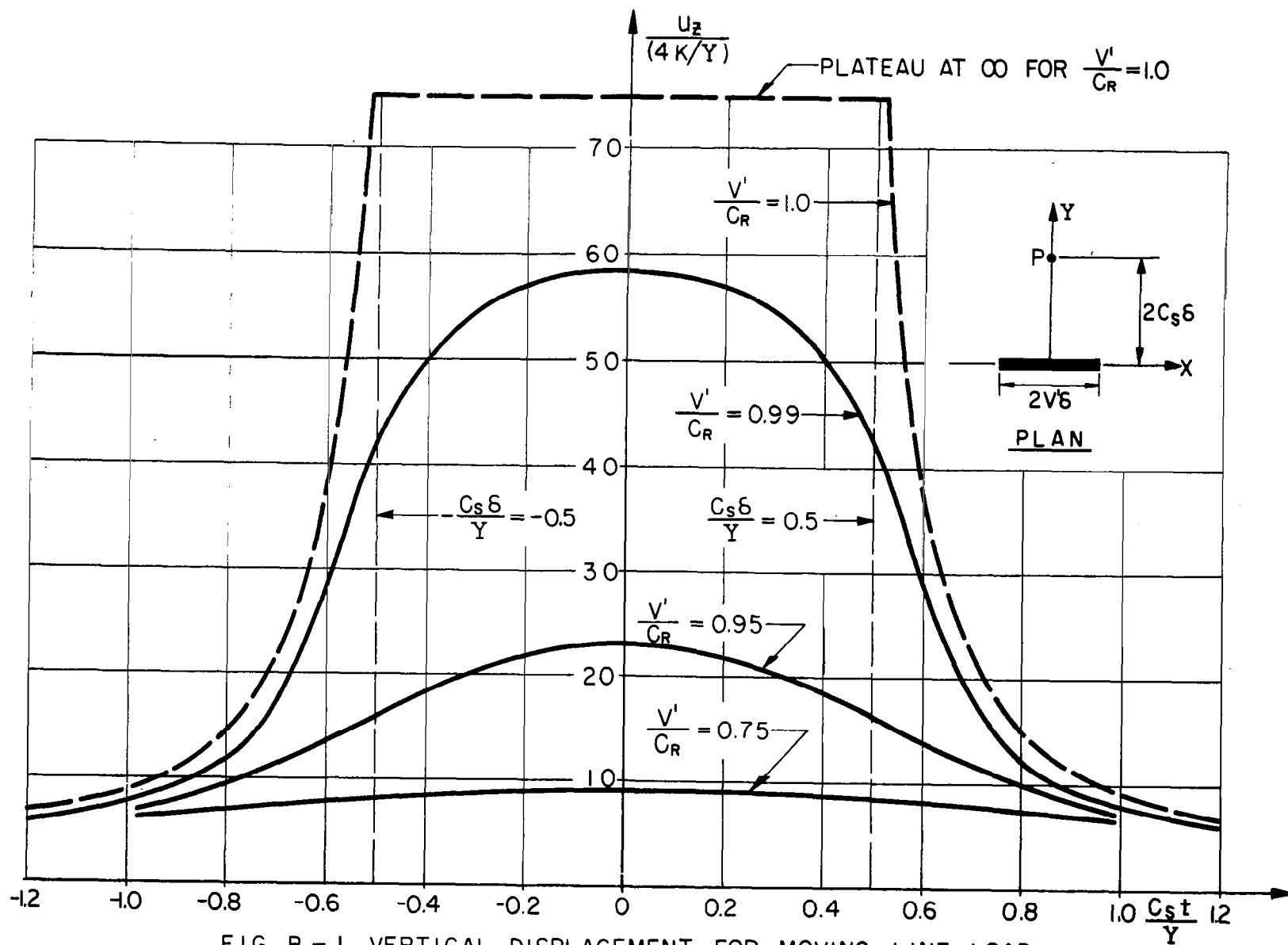


FIG. B - 1 VERTICAL DISPLACEMENT FOR MOVING LINE LOAD

CASE A ;  $V' \leq C_R$  ;  $V = 1/4$  ;  $C_s \delta / Y = 0.5$

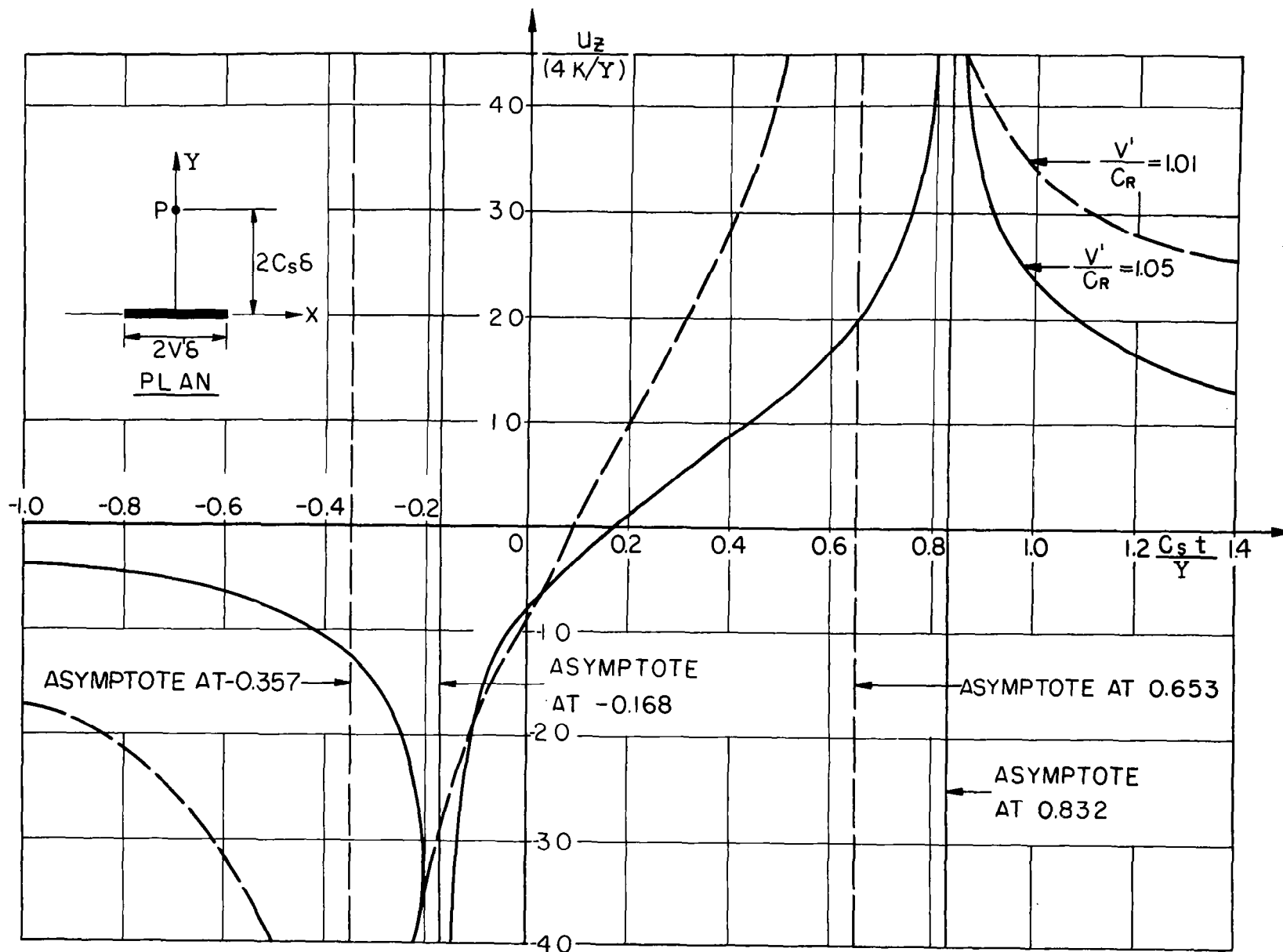


FIG. B-2 VERTICAL DISPLACEMENT FOR MOVING LINE LOAD

CASE B-  $C_R < V' < C_S$ ;  $V = 1/4$ ;  $C_s\delta/Y = 0.5$

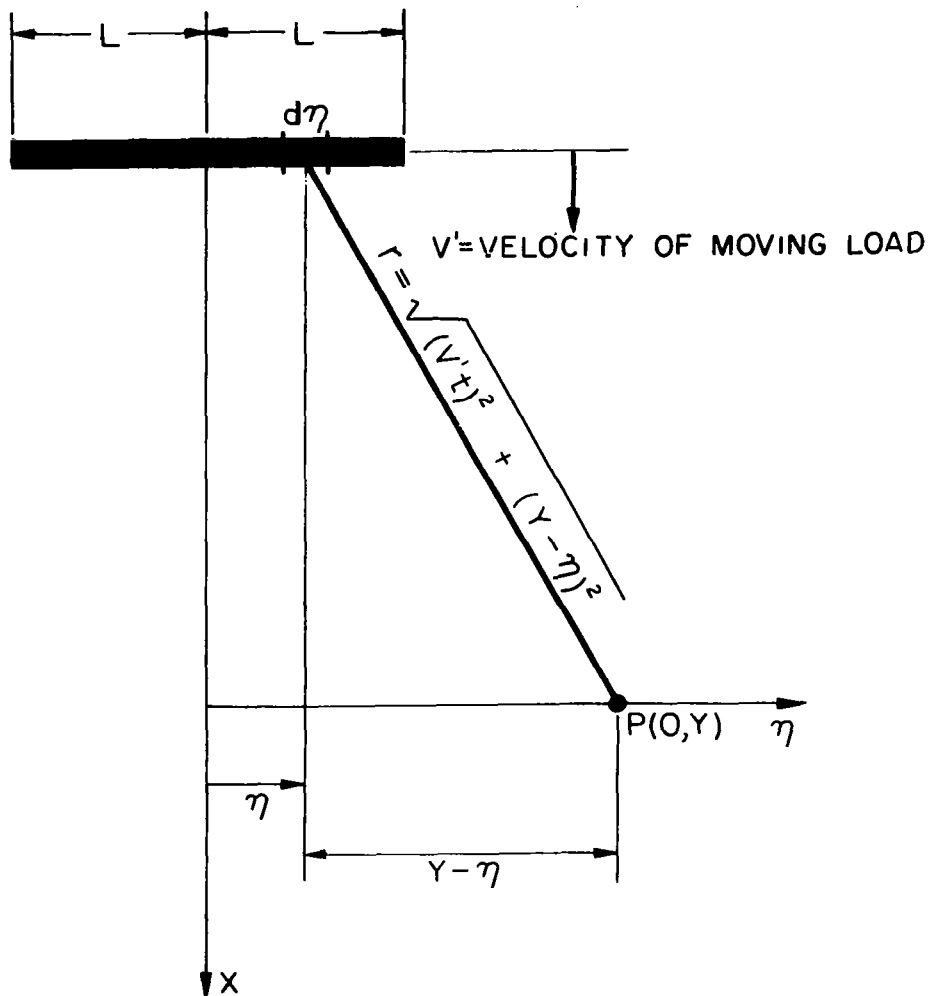


FIG. B-3 PLAN VIEW OF MOVING  
TRANSVERSE LINE LOAD